Fluids Notes

1) Describing fluids

- Continuum hypothesis: ignore molecular structure and treat fluids as a continuum. Need $\lambda_{mfp} \ll L$
- Lagrangian / Eulerian description
- Material derivative: track fluid parcels and observe fluid property φ:

$$\frac{D\varphi}{Dt} := \frac{d}{dt}\left(\varphi(r(t),t)\right) = \frac{\partial \varphi}{\partial t} + \frac{dr}{dt}\frac{d\varphi}{dr} = \frac{\partial \varphi}{\partial t} + v \cdot \nabla \varphi .$$

  - Relates Lagrangian rates of change to changes in the Eulerian field.

- Convection theorem: (Reynold's transport theorem).

$$\frac{d}{dt}\int_{\Omega(t)} f(x,t)\, d\vec{x} = \int_{\Omega(t)} \frac{\partial f}{\partial t}\, d\vec{x} + \int_{\partial\Omega(t)} f \vec{v}\cdot\vec{n}\, dA$$

- Useful identities to remember:

1) $\nabla \cdot (\nabla \times \vec{F}) = 0$
2) $\nabla \times (\nabla f) = 0$
3) $\nabla\left(\frac{u^2}{2}\right) = u \cdot \nabla u + u \times \nabla \times u$
4) $\nabla \times (\vec{F} \times \vec{G}) = \vec{F}(\nabla \cdot \vec{G}) - \vec{G}(\nabla \cdot \vec{F}) + (\vec{G} \cdot \nabla)\vec{F} - (\vec{F} \cdot \nabla)\vec{G}$
5) $\nabla^2 \vec{F} = \nabla(\nabla \cdot \vec{F}) - \nabla \times (\nabla \times \vec{F})$.

- Fundamental vector calc theorems:

  * $\oint_C \nabla f \cdot d\vec{r} = 0$, $\int_{\vec{a}}^{\vec{b}} \nabla f \cdot d\vec{r} = f(\vec{b}) - f(\vec{a})$ Gradient theorem
  * $\iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, dA = \oint_{\partial S} \vec{F} \cdot dr$ Stokes' theorem
  * $\iiint_V \nabla \cdot \vec{F}\, dV = \iint_{\partial V} \vec{F} \cdot \hat{n}\, dA$ Gauss' theorem / Divergence theorem.

2) Continuity and Momentum equation

Continuity

$$\frac{D}{Dt}(\rho \Delta V) = \Delta V \frac{D\rho}{Dt} + \rho \frac{D\Delta V}{Dt}$$

$$= \Delta V \frac{D\rho}{Dt} + \rho \, \nabla\cdot\vec{v} \, \Delta V$$

$$= \left(\frac{\partial \rho}{\partial t} + \vec{v}\cdot\nabla\rho + \rho\nabla\cdot\vec{v}\right)\Delta V = \left(\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{v})\right)\Delta V = 0$$

$$\Rightarrow \boxed{\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{v}) = 0}$$

~~Momentum~~

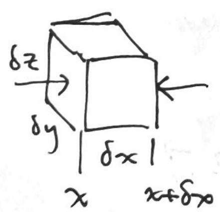
- Incompressibility:

$$\int_\Omega \frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{v})\, dV = \int_\Omega \frac{\partial \rho}{\partial t}\, dV + \int_{\partial\Omega} \rho\vec{v}\cdot\vec{n}\, dA$$

$$\rho = \rho_0 \Rightarrow \int_{\partial\Omega} \vec{v}\cdot n \, dA = 0$$

Momentum

- Ideal fluids:



$$ma = F$$

$$\rho \frac{D\vec{v}}{Dt}\Delta V = \delta y\,\delta z\,(p(x) - p(x+\delta x))$$

$$\rightsquigarrow \rho \frac{D\vec{v}}{Dt} = -\frac{\partial p}{\partial x}$$

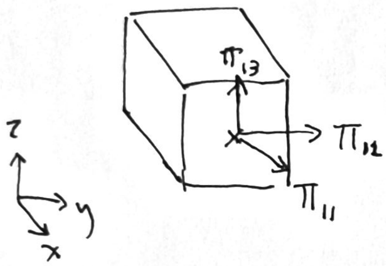
$$\rightsquigarrow \rho\left(\frac{\partial \vec{v}}{\partial t} + \vec{v}\cdot\nabla\vec{v}\right) = -\nabla p + \rho\vec{g}$$

- Hydrostatic balance:

$$0 = -\frac{\partial p}{\partial z} - \rho g$$

- <u>Momentum with stress tensor</u>

$$\frac{D}{Dt}\left(\rho \vec{v}\, dV\right) = \rho\, dV \vec{g} + \sum \vec{F}_{surf}$$



Define $\overleftrightarrow{\Pi}$, Π_{ij}

i = normal vector to plane

j = direction of force

Surface given by $d\vec{A} = \sum_{i=1}^{3} n_i e_i \, dA$

Force in direction $\vec{e}_j$: $F_j \, dA = \sum_{i=1}^{3} \Pi_{ij} n_i \, dA$

$$\Rightarrow \quad \vec{F}_{surf} = \vec{n} \cdot \overleftrightarrow{\Pi}$$

Total contribution over whole surface:

$$\sum \vec{F}_{surf} = \int_{\partial\Omega} \vec{n} \cdot \overleftrightarrow{\Pi} \, dA = \int_{\Omega} \nabla \cdot \overleftrightarrow{\Pi} \, dV$$

$$\Rightarrow \cancel{\vec{v}\,\frac{D\rho\Delta V}{Dt}} + \rho \frac{D\vec{v}}{Dt}\Delta V = \rho \Delta V \vec{g} + \nabla \cdot \overleftrightarrow{\Pi}\, \Delta V.$$

$$\Rightarrow \quad \rho \frac{D\vec{v}}{Dt} = \rho \vec{g} + \nabla \cdot \overleftrightarrow{\Pi}$$

$$\boxed{\rho \frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} = \rho \vec{g} + \nabla \cdot \overleftrightarrow{\Pi}}$$

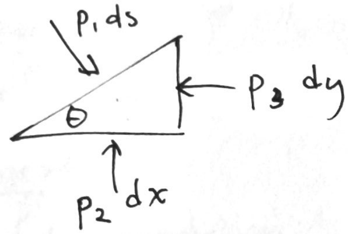
$$= \rho \vec{g} - \nabla p + \mu \nabla^2 \vec{v}$$

Newtonian Fluids

$$\overleftrightarrow{\tau} = \mu \nabla \vec{v}$$

3) Ideal Fluids

- Pressure acts isotropically:



$\rightsquigarrow \; p_1 \sin\theta \, ds = p_3 \, dy$

$\rightsquigarrow \; p_2 \, dx = p_1 \cos\theta \, ds$

$ds = \dfrac{dx}{\cos\theta} \Rightarrow p_1 = p_2$

$ds = \dfrac{dy}{\sin\theta} \Rightarrow p_1 = p_3$

- Ideal fluid equations:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$$

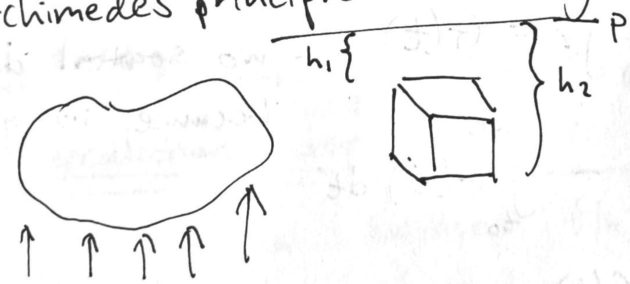
$$\rho \left(\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} \right) = -\nabla p + \rho \vec{g}$$

Euler Equations

- Hydrostatic balance:
  - Pascal's law: isobars in a resting fluid are horizontal: all fluid points at the same depth are at the same pressure.

$$\frac{dp}{dz} = -\rho g$$

  - Archimedes principle and buoyancy:



$$\sum F_p = (p_0 + \rho_E g h_2) A - (p_0 + \rho_E g h_1) A = \rho g (h_2 - h_1) A = \rho_E g V$$

$\rightarrow$ buoyancy force equals displaced volume.

- **Bernoulli's theorem**
  - Take steady, ideal (inviscid), constant density flow:

$$\rho_0(\vec{u}\cdot\nabla\vec{v}) = -\nabla p + \rho_0\vec{g}$$

$$\underbrace{\vec{v}\cdot\nabla\vec{v}} = -\nabla\left(\frac{p}{\rho_0} + gz\right)$$

$$\nabla\left(\frac{\|v\|^2}{2}\right) - \vec{v}\times(\nabla\times\vec{v}) = -\nabla\left(\frac{p}{\rho_0} + gz\right)$$

$$\rightsquigarrow \nabla\left(\frac{p}{\rho_0} + \frac{\|v\|^2}{2} + gz\right) = v\times(\nabla\times\vec{v})$$

$$\rightsquigarrow \boxed{\vec{v}\cdot\nabla B = 0 \implies B = \frac{p}{\rho_0} + \frac{\|v\|^2}{2} + gz \text{ constant along streamlines}}$$

- **Bernoulli's theorem for unsteady ~~...~~ potential flows:**
  - Suppose $\vec{v} = \nabla\phi$

$$\rightsquigarrow \nabla\times\vec{v} = \nabla\times(\nabla\phi) = 0.$$

we get $\frac{\partial\vec{v}}{\partial t} + \nabla B = v\times(\nabla\times\vec{v}) = 0$ since $\vec{\omega} = 0$

$$\rightsquigarrow \frac{\partial}{\partial t}\nabla\phi + \nabla B = 0$$

$$\rightsquigarrow \nabla\left(\frac{\partial\phi}{\partial t} + \frac{p}{\rho_0} + \frac{\|v\|^2}{2} + gz\right) = 0$$

$$\Rightarrow \frac{\partial\phi}{\partial t} + \frac{p}{\rho_0} + \frac{\|v\|^2}{2} + gz = G(t)$$

← no spatial dependence because no gradient.

If we defined $\phi' = \phi - \int_{t_0}^{t} G(t')\,dt'$

then $\frac{\partial\phi}{\partial t} = \frac{\partial\phi'}{\partial t} + G(t)$

So we get $\vec{v} = \nabla\phi'$, such that

$$\frac{\partial\phi'}{\partial t} + \frac{p}{\rho_0} + \frac{\|v\|^2}{2} + gz = 0.$$

4) Vorticity

- Definition:

$$\vec{\omega} = \nabla \times \vec{v}$$

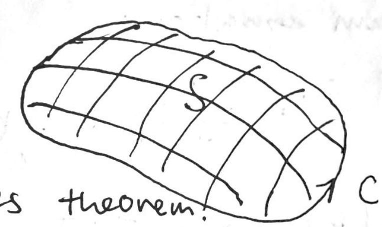
- For $\vec{u} = (u,v)$, $\vec{\omega} = \nabla \times \vec{u} = (v_x - u_y)\hat{k} = \zeta \hat{k}$.

- Examples:
  - Rigid body rotation: $\vec{u} = (-y, x) \Rightarrow \vec{\omega} = \langle 0,0,2 \rangle$.
  - Shear flow: $\vec{u} = (y, 0) \Rightarrow \vec{\omega} = (0,0,-1)$
  - Line vortex flow: $\vec{u} = \frac{\vec{e}_\theta}{r} = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right) \Rightarrow \vec{\omega} = \begin{cases} \vec{0}, & (x,y) \neq 0 \\ \text{undefined at origin.} \end{cases}$

- Circulation

$$\Gamma = \oint_C \vec{u} \cdot d\vec{r} = \iint_S \nabla \times \vec{u} \, dA$$

$$= \iint_S \omega \cdot n \, dA \quad \text{via Stokes theorem!}$$



- Examples:

* Point vortex: If $\vec{u} = \frac{1}{2\pi r}\vec{e}_\theta$, then on ~~a loop~~ the unit circle

$$\Gamma = \oint_C u \cdot dr = \frac{1}{2\pi}\int \frac{\vec{e}_\theta}{r} \cdot d\vec{r} = \frac{1}{2\pi}\int_0^{2\pi} 1 \, dr = 1$$

So $\vec{u} = \frac{1}{2\pi r}\vec{e}_\theta$ is often called the point vortex of unit strength

(and more generally $\vec{u} = \frac{\Gamma_0}{2\pi r}\vec{e}_\theta$ is a point vortex of strength Γ_0.

- Kelvin's circulation theorem

> Theorem: For ideal, barotropic fluids $(p = p(\rho))$,
>
> $$\frac{d}{dt}\Gamma_{C(t)} = 0,$$
>
> where $C(t)$ is a material curve and $\Gamma_{C(t)}$ is its circulation.

- Proof of Kelvin's circulation theorem:

  * Parameterize $C(t) = \mathbf{r}(t; s)$ $a \le s \le b$. Then

$$\frac{d}{dt}\Gamma_{C(t)} = \frac{D}{Dt}\oint_{C(t)} u \cdot dr = \frac{D}{Dt}\int_{s=a}^{s=b} \vec{u} \cdot \frac{\partial r}{\partial s}\, ds$$

  * Integral parameterization independent of time, take deriv inside

$$= \int_a^b \frac{D}{Dt}\left(u \cdot \frac{\partial r}{\partial s}\right) ds = \int \frac{\partial r}{\partial s} \cdot \frac{D\vec{u}}{Dt} + u \frac{\partial}{\partial s}\left(\frac{Dr}{Dt}\right) ds$$

$$= \int_a^b \frac{\partial r}{\partial s} \cdot \frac{Du}{Dt} + u \frac{\partial u}{\partial s}\, ds = \int_a^b \frac{\partial r}{\partial s} \cdot \frac{Du}{Dt}\, ds + \underbrace{\int_a^b \frac{\partial}{\partial s}\left(\frac{|u|^2}{2}\right) ds}_{=0}$$

  * Input momentum equation:

$$= \int_a^b \frac{\partial r}{\partial s} \cdot \left(\frac{-1}{\rho}\nabla p + \nabla \chi\right) ds$$

  * Define $N(p)$ such that $\nabla N = \frac{1}{\rho}\nabla p$

Try $N(p) = \int \frac{1}{\rho}\frac{dp}{d\rho}\, d\rho$

So $\nabla N = \frac{dN}{d\rho}\nabla \rho = \frac{1}{\rho}\frac{dp}{d\rho}\nabla \rho = \frac{1}{\rho}\nabla p$

  * Gradient integral over closed curve is zero:

$$\frac{d\Gamma_{C(t)}}{dt} = \int_a^b \frac{\partial r}{\partial s} \cdot \left(\nabla(\chi - N)\right) ds$$

$$= \int_{C(t)} \nabla(\chi - N) \cdot d\vec{r} = 0.$$

- Note: extension for ~~geostrophic~~ rotating flows: ~~$\frac{Du}{Dt}$ $u_r = \vec{u} + \Omega \times \vec{r}$~~

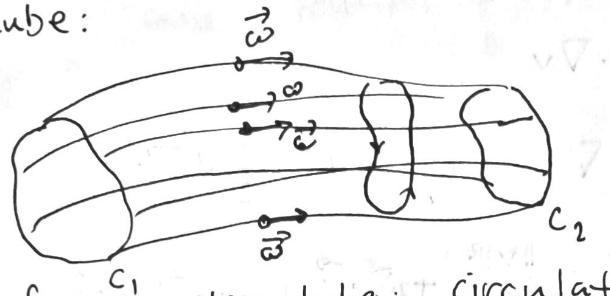
$$\frac{D\vec{v}}{Dt} = -2\Omega \times \vec{v} - \frac{1}{\rho}\nabla p + \nabla \chi$$

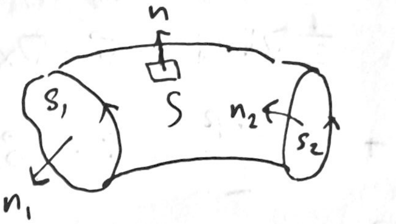
$$\rightsquigarrow \int_a^b \frac{\partial r}{\partial s} \cdot \left(-2\Omega \times \vec{v} + \nabla(\chi - N)\right) ds$$

- Helmholtz laws

- Vortex lines: "flow" lines of the vorticity field: $\frac{dx}{\omega_1} = \frac{dy}{\omega_2} = \frac{dz}{\omega_3}$

- Vortex tube:



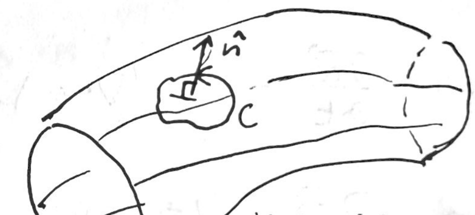


- Strength of a vortex tube: circulation on the surface of a vortex tube around the tube once.

  * Independent of loop: $\oint_{C_1} \vec{u}\cdot d\vec{r} - \oint_{C_2} \vec{u}\cdot d\vec{r} = \iint_{S_1} \vec{\omega}\cdot\vec{n}\, dA - \iint_{S_2} \vec{\omega}\cdot\vec{n}\, dA$

  $$= \iint_{S_1} \vec{\omega}\cdot\vec{n}_1 dA - \iint_{S_2} \vec{\omega}\cdot\vec{n}_2 dA + \iint_{S} \vec{\omega}\cdot\vec{n}\, dA = \iiint_V \nabla\cdot\vec{\omega}\, dV = 0$$

- Helmholtz laws:



H. law (1)

1) Vortex lines are material curves:

   $\Gamma_{C(t_1)} = \int_C u\cdot dr = \iint_S \omega\cdot n\, dA = 0$

   $\frac{d\Gamma_C}{dt} = 0 \Rightarrow \Gamma_{C(t_2)} = 0 = \iint_{S'} \omega\cdot n\, dA$,

   so ω is still tangential to S' and we are on the same vortex tubes.

   * $\Rightarrow$ Vortex tubes move with the fluid, contain fixed mass.

2) Fluid elements free of vorticity remain free of vorticity:

   $\omega = 0 \Rightarrow \Gamma_C = \iint_S \vec{\omega}\cdot\vec{n} = 0$

3) Vortex tubes cannot end within the fluid.

4) Vortex strength is constant.

- Vortex stretching:

  * Mass inside a vortex tube is fixed: $\rho_1 l_1 \Delta S_1 = \rho_2 l_2 \Delta S_2$
  * Vortex strength is constant: $\omega_1 \Delta S_1 = \omega_2 \Delta S_2$

  $$\Rightarrow \frac{\omega_2}{\omega_1} = \frac{l_2}{l_1} \quad (\rho_1 = \rho_2)$$

  stretching a vortex tube intensifies the vorticity.

- The vorticity equation

$$\frac{\partial \vec{v}}{\partial t} + \underbrace{\vec{v}\cdot\nabla\vec{v}} = \frac{-1}{\rho}\nabla p + \nabla \chi + \nu \nabla^2 \vec{v}$$

$$\frac{1}{2}\nabla\|v\|^2 - \vec{v}\times\nabla\times\vec{v} = v\cdot\nabla v$$

$$\Leftrightarrow \frac{\partial \vec{v}}{\partial t} + \nabla\frac{\|v\|^2}{2} - \vec{v}\times\vec{\omega} = \frac{-1}{\rho}\nabla p + \nabla\chi + \nu\left[\nabla(\nabla\cdot\vec{v}) - \nabla\times(\nabla\times\vec{v})\right]$$

$$\Rightarrow \frac{\partial \vec{v}}{\partial t} - \vec{v}\times\vec{\omega} = \nabla\left(\frac{-p}{\rho} + \chi - \frac{\|v\|^2}{2} + \nu\nabla\cdot\vec{v}\right) - \nu\nabla\times\vec{\omega}$$

Assuming constant density.

Use identity: $\nabla\times(A\times B) = A(\nabla\cdot B) - B(\nabla\cdot A) + (B\cdot\nabla)A - (A\cdot\nabla)B$
and take the curl:

$$\Rightarrow \frac{\partial\vec{\omega}}{\partial t} - \left[v(\nabla\cdot\omega) - \omega\nabla\cdot v + (\omega\cdot\nabla)\vec{v} - \vec{v}\cdot\nabla\omega\right] = -\nu\nabla\times(\nabla\times\omega)$$

$$\Rightarrow \frac{D\vec{\omega}}{Dt} = (\vec{\omega}\cdot\nabla)\vec{v} - \nu\nabla\times(\nabla\times\omega)$$

Use $\nabla^2\omega = \nabla(\nabla\cdot\omega) - \nabla\times(\nabla\times\omega)$

$$\Rightarrow \boxed{\frac{D\vec{\omega}}{Dt} = (\vec{\omega}\cdot\nabla)\vec{v} + \nu\nabla^2\vec{\omega}}$$

Vorticity equation for constant density flows

5) Potential Flow

- Definition and Examples.

* $\vec{u} = \nabla\phi$ for some potential $\phi(r,t) = \int_\gamma u(\vec{x},t)\cdot d\vec{x}$.

* Potential flows are irrotational:

$$\nabla \times \vec{v} = \nabla \times (\nabla\phi) = 0 .$$

- Examples:
  - * point vortex flow: $\vec{u} = \frac{\Gamma}{2\pi r}\hat{e}_\theta$

$$\phi = \frac{\Gamma\theta}{2\pi} \Rightarrow \nabla\phi = \frac{\partial\phi}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{e}_\theta = \frac{\Gamma}{2\pi r}\hat{e}_\theta$$

  - * Point source flow:

If $\vec{v}$ is incompressible, $\nabla\cdot\vec{v} = \nabla^2\phi = 0$

Fundamental solution of Laplace's equation in 2D is

$$\phi(r,\theta) = \frac{Q}{2\pi}\ln r .$$

$$\Rightarrow \vec{v} = \nabla\phi = \frac{Q}{2\pi r}\hat{e}_r$$

  - * Superpositions: E.g., point source in a uniform flow:

$$\phi_{tot} = \phi_{point} + \phi_{unif} = \frac{Q}{2\pi}\ln r + U r\cos\alpha$$

- Stream functions (2D flow).
  - Require incompressibility: $\nabla\cdot\vec{u} = 0$
  - Define $\psi(\vec{r}) = \int_\gamma \vec{u}\cdot\hat{n}\,ds = \int u\,dy - v\,dx$

$$\Rightarrow \boxed{u = \frac{\partial\psi}{\partial y} \qquad v = \frac{-\partial\psi}{\partial x}}$$

  - Note $\vec{u}\cdot\nabla\psi = \frac{\partial\psi}{\partial y}\frac{\partial\psi}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\psi}{\partial y} = 0$

    Contours of stream functions are streamlines

  - Can write $\vec{u} = \nabla\psi \times \hat{k}$.

• Incompressible, irrotational potential flow

- If $\nabla \times \vec{u} = 0$, then $\vec{u} = \nabla \phi$. $\nabla \cdot \vec{u} = 0$ implies $\nabla^2 \phi = 0$.
- similarly, if ~~$\nabla \times \vec{u} = 0$, $\nabla \times u = \nabla \times$~~ $\nabla \cdot \vec{u} = 0$, $\vec{u} = \nabla \psi \times \hat{k}$
- If $\nabla \times \vec{u} = 0$ then $\nabla \times (\nabla \psi \times \hat{k}) = -\hat{k}\, \nabla^2 \psi = \vec{0}$.

So for irrotational incompressible 2D flow,

$$\boxed{\begin{aligned} \nabla^2 \phi &= 0 \\ \nabla^2 \psi &= 0 \end{aligned}}$$

• The complex potential

- Suppose irrotational, incompressible 2D flow.

$$\vec{u} = (u, v) = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}\right) = \left(\frac{\partial \psi}{\partial y}, \frac{-\partial \psi}{\partial x}\right)$$

$$\Rightarrow \quad \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}; \quad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \qquad \text{Cauchy Riemann equations.}$$

- The complex potential

$$w(z) = \phi(x,y) + i\,\psi(x,y)$$

is holomorphic.

* The derivative is

$$\frac{dw}{dz} = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial y} = u - iv$$

* The speed is

$$\|\vec{u}\| = \sqrt{u^2 + v^2} = \left|\frac{dw}{dz}\right|$$

Point Vortex Flow

$$\vec{u} = \frac{\Gamma}{2\pi r}\hat{e}_\theta$$

$$\phi = \frac{\Gamma \theta}{2\pi}$$

$$\psi = \frac{-\Gamma}{2\pi}\log(r)$$

Point Source Flow

$$\vec{u} = \frac{Q}{2\pi r}\hat{e}_r$$

$$\phi = \frac{Q}{2\pi}\ln(r)$$

$$\psi = \frac{Q\theta}{2\pi}$$ I think

- Flow around a cylinder

- Milne-Thomson Circle theorem:

  * Gives the means for finding the flow around a cylinder.

  * If $f(z)$ is the complex potential for some flow, with some singularities outside the circle $|z| = a$, then

$$w(z) = f(z) + \overline{f\left(\frac{a^2}{\bar{z}}\right)}$$

has the same singularities but with $|z| = a$ as a streamline.

- Uniform flow around a cylinder of radius R:

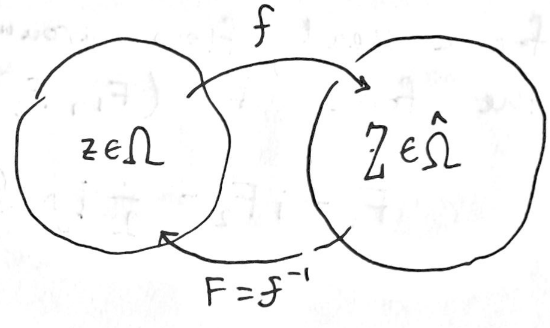
$$f(z) = Uz$$

$$\rightsquigarrow \; w(z) = U\left(z + \frac{R^2}{z}\right).$$

6 Lift and Drag

- **Conformal Mapping for 2D potential flow**
  - If $f: \Omega \to \hat{\Omega}$ is conformal (biholomorphic) map, and $w(z) = \phi(x,y) + i\psi(x,y)$ is defined on Ω we can define a complex potential on $\hat{\Omega}$ by $W(Z) = w(f^{-1}(Z))$

    $= \Phi(X,Y) + i\Psi(X,Y)$



  - streamlines map to streamlines, ~~potential to~~ equipotentials to equipotentials.

- **Forces on a cylinder immersed in potential flow**
  - Complex potential for flow around cylinder, with circulation:

$$w(z) = U\left(z + \frac{R^2}{z}\right) - \frac{i\Gamma}{2\pi}\ln z$$

$$\rightsquigarrow\ u_r = U\left(1 - \frac{R^2}{r^2}\right)\cos\theta$$

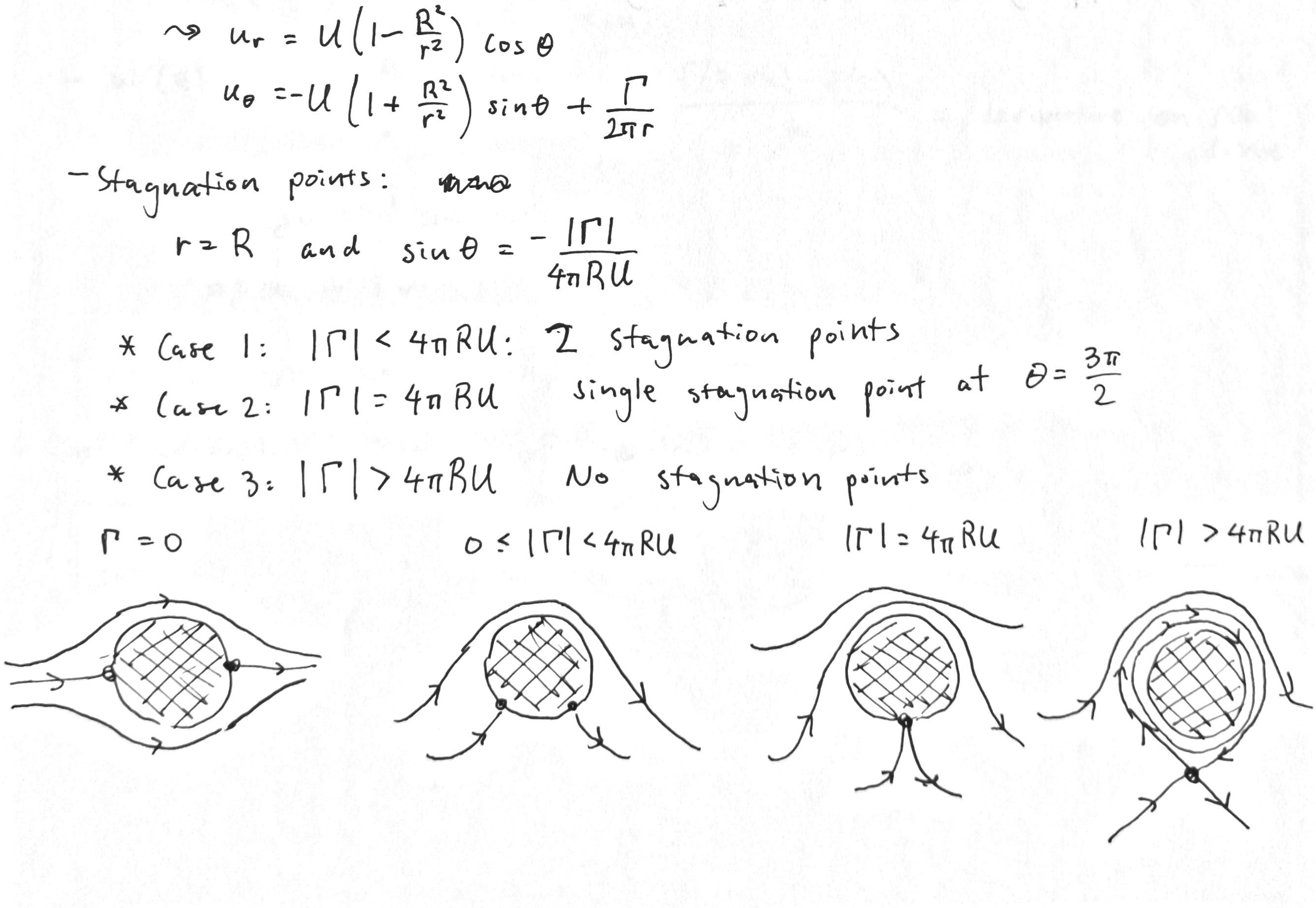
$$u_\theta = -U\left(1 + \frac{R^2}{r^2}\right)\sin\theta + \frac{\Gamma}{2\pi r}$$

  - Stagnation points:

$$r = R \quad \text{and} \quad \sin\theta = -\frac{|\Gamma|}{4\pi R U}$$

    * Case 1: $|\Gamma| < 4\pi RU$: 2 stagnation points
    * Case 2: $|\Gamma| = 4\pi RU$ single stagnation point at $\theta = \frac{3\pi}{2}$
    * Case 3: $|\Gamma| > 4\pi RU$ No stagnation points

$\Gamma = 0$ $\qquad$ $0 \le |\Gamma| < 4\pi RU$ $\qquad$ $|\Gamma| = 4\pi RU$ $\qquad$ $|\Gamma| > 4\pi RU$



- Lift force
  * Can use Bernoulli theorem to compute pressure on cylinder
  * Lift force is $L = -\rho U \Gamma$

• Kutta-Joukowski theorem

- Blasius' Theorem: If $w(z)$ is the complex potential for a steady flow around a fixed body, then the force $\vec{F} = (F_1, F_2)$ is given by

$$F_1 - i F_2 = \frac{1}{2} i \rho \oint_C \left(\frac{dw}{dz}\right)^2 dz$$

- Kutta-Joukowski: consider uniform flow $\vec{u} = (U \cos\alpha, U \sin\alpha)$ around a closed body with circulation Γ. Then

$$F_1 = \rho U L \sin\alpha, \quad F_2 = -\rho U L \cos\alpha.$$

Complex Potential Flow

Complex Potential

- Irrotational 2D flow: define potential ϕ

  $\vec{u} = \nabla\phi$ $\quad u = \phi_x$, $\quad v = \phi_y$

- Incompressible 2D flow: define streamfunction ψ:

  $\vec{u} = \nabla\psi \times \hat{k}$ $\quad u = \psi_y$, $\quad v = -\psi_x$

- Incompressible / irrotational 2D flow:

  $\Rightarrow$

  $$\phi_x = \psi_y$$

  $$\phi_y = -\psi_x$$

  Cauchy Riemann equations.

- can define complex potential

  $$w(z) = \phi(x,y) + i\,\psi(x,y)$$

  - $w'(z) = \lim_{z'\to z} \frac{f(z') - f(z)}{z' - z} = \lim_{k\to 0} \frac{f(z+k) - f(z)}{k}$ = derivative on real line

    $$= \frac{\partial\phi}{\partial x} + i\frac{\partial\psi}{\partial x}$$

    $$= u - iv$$

  - $|w'(z)| = \sqrt{u^2 + v^2} = \|\vec{u}\|$.

- Can use all the theory of complex analysis now.

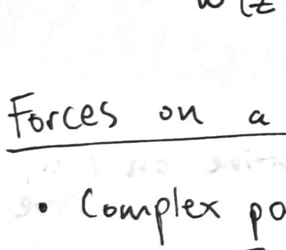
Milne Thomson circle theorem / flow around cylinders

- Milne Thomson theorem: If $f(z)$ is a complex potential where all singularities are in $|z| > a$, then
  $$w(z) = f(z) + \overline{f\left(\frac{a^2}{\bar{z}}\right)}$$ has all its singularities shared with $f(z)$, but circle $|z| = a$ is a stream line.

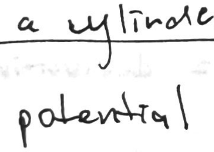
- Flow around cylinder:
  - Infinitely long cylinder of radius R in uniform fluid field $u = \langle U, 0 \rangle$
    - * u has potential $f(z) = Uz$
    - * Add cylinder: singularities stay the same, but streamline occurs at $|z| = R$: Apply Milne-Thomson theorem.
  $$w(z) = Uz + \overline{U\frac{R^2}{\bar{z}}} = Uz + U\frac{R^2}{z} = U\left(z + \frac{R^2}{z}\right)$$

Forces on a cylinder

- Complex potential for point vortex:
  $$u = \frac{\Gamma}{2\pi r}\hat{e}_\theta$$
  - Potential: $\vec{u} = \nabla\phi = \frac{\partial\phi}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{e}_\theta$
  $$\Rightarrow \frac{\partial\phi}{\partial\theta} = \frac{\Gamma}{2\pi} \Rightarrow \boxed{\phi = \frac{\Gamma\theta}{2\pi}}$$
  - Streamfunction:
  $$\psi = \int \vec{u}\cdot\hat{n}\, d\ell \qquad \vec{u} = u_r\hat{e}_r + u_\theta\hat{e}_\theta$$
  ~~$\vec{u} = \nabla\times(\psi\hat{k})$~~



$$= \int u_\theta\hat{e}_\theta\cdot\hat{n}\, d\ell = \cancel{\int\frac{\Gamma}{2\pi r}d\theta} = \int\frac{\Gamma}{2\pi r}dr$$



$$= \frac{-\Gamma}{2\pi}\log r$$

  - Complex potential: $$\boxed{w(z) = \frac{-i\Gamma}{2\pi}\ln(z)}$$

- Because the point vortex flow has circular streamlines superimposing it to the cylinder flow preserves the streamline, but adds circulation:

$$w(z) = U\left(z + \frac{R^2}{z}\right) - \frac{i\Gamma}{2\pi}\log(z)$$

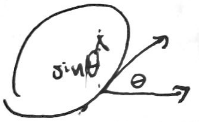
- Can compute stagnation points, where $\vec{u} = 0$.

$$w(re^{i\theta}) = U r(\cos\theta + i\sin\theta) + R^2(\cos\theta - i\sin\theta) + \frac{\Gamma}{2\pi}(\theta - i\ln r)$$

$$\rightarrow \phi(r,\theta),\ \psi(r,\theta).$$

- Use Bernoulli theorem on streamline tangent to cylinder.
- Symmetry of stagnation points $\Rightarrow$ vertical net force only possible.

$$dF = -p R \sin\theta \, d\theta$$



$$\text{lift} = \int -p R \sin\theta \, d\theta$$

(p given by Bernoulli)

$$\rightarrow \boxed{L = -\rho U \Gamma}$$

Blasius' and Kutta-Joukowski

- Blasius theorem: force around a body with contour C in field $w(z)$

$$F_x - iF_y = \frac{1}{2} i\rho \oint_C \left|\frac{dw}{dz}\right|^2 dz$$

- Kutta-Joukowski: lift around any simple closed curve; immersed in uniform flow $f(z) = Uz$

$$\boxed{\begin{aligned} F_1 &= 0 \\ F_2 &= -\rho U \Gamma \end{aligned}}$$

7 Incompressible fluid waves

Terminology

$$\eta = A\cos\left[\frac{2\pi}{\lambda}(x - ct)\right] \qquad \lambda = \text{wavelength}$$

- $k = \frac{2\pi}{\lambda}$ wave number
- A amplitude
- c: phase speed

Linearization

- Model examples: shallow water equations

$$\frac{\partial \eta}{\partial t} + \nabla\cdot(hu) = 0$$

$$\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u} + \vec{f}\times\vec{u} = -g\nabla\eta$$

- Mean-eddy decomposition:

$$u = \bar{u} + u' \qquad h = H + \eta$$

$$\frac{\partial \eta}{\partial t} + u\cdot\nabla h + h\nabla\cdot u = 0$$

becomes

$$\frac{\partial \eta}{\partial t} + (\bar{u} + u')\nabla\eta + (H + \eta)\nabla\cdot(\bar{u} + u')$$

$$\approx \frac{\partial \eta}{\partial t} + \bar{u}\nabla\eta - H\nabla\cdot\vec{u} = 0$$

- Linearization about a mean state $\bar{u} = 0$ gives

$$\frac{\partial \eta}{\partial t} + H\nabla\cdot u = 0$$

$$\frac{\partial u}{\partial t} - fv = -g\eta_x$$

$$\frac{\partial v}{\partial t} + fu = -g\eta_y$$

- Since derivatives of linear compos of exponentials are also exponentials it make sense to search for exponential solutions

$$\eta = \hat{\eta} e^{i(kx-\omega t)} \quad u = \hat{u} e^{i(kx-\omega t)} \quad v = \hat{v} e^{i(kx-\omega t)}$$

one way to do this is to consider $\vec{u} = \begin{bmatrix} \eta \\ u \\ v \end{bmatrix}$

Our system is

$$\eta_t + H(u_x + v_y) = 0$$

$$u_t - fv + g\eta_x = 0$$

$$v_t + fu + g\eta_y = 0$$

$$\rightsquigarrow \begin{bmatrix} -i\omega & ikH & ilH \\ ikg & -i\omega & -f \\ ilg & f & -i\omega \end{bmatrix} \begin{bmatrix} \eta \\ u \\ v \end{bmatrix} = 0 \quad = \quad Au = 0$$

need $\det(A) = 0$:

$$0 = -i\omega(-\omega^2 + f^2) - ikH(\omega kg + ilfg) + ilH(ikfg - l\omega g)$$

$$\Rightarrow -\omega(-\omega^2 + f^2) + H(-\omega k^2 g - iklfg + iklfg - \omega l^2 g)$$

$$\Rightarrow \omega(\omega^2 - f^2 - Hg(k^2 + l^2))$$

$$\Rightarrow \boxed{\omega^2 = f_0^2 + gH(k^2 + l^2)}.$$

8 Compressible Fluids

Thermodynamic equation

- Without incompressibility, we need another equation to close the system. We can try using an equation of state to show how quantities are related:

$$p = \rho R T \quad \text{ideal gas law}$$

$$\rho = \rho_0 \left[1 - \beta_T (T - T_0) + \beta_p (p - p_0) + \beta_S (S - S_0)\right] \quad \text{ocean EOS}$$

But this introduces new thermodynamic variables. Need more equations

- For the ocean, we use empirical thermodynamic equation

$$\frac{D\theta}{Dt} = \dot{Q}_T - \dot{F}_T$$

$$\frac{DS}{Dt} = \dot{Q}_T - \dot{F}_T$$

- For gases, we can use the first law of thermodynamics:

$$dU = -p\, d\alpha + \delta Q^{m}$$

$$\Rightarrow \quad \frac{DU}{Dt} + p \frac{D\alpha}{Dt} = \dot{Q} \qquad p\frac{D\alpha}{Dt} = p \frac{D}{Dt}\left(\frac{1}{\rho}\right) = \frac{-p}{\rho^2}\frac{D\rho}{Dt}$$

$$\Rightarrow \quad \dot{Q} = \frac{DU}{Dt} - \frac{p}{\rho^2}\frac{D\rho}{Dt} = \frac{DU}{Dt} - \frac{p}{\rho^2}\left(-\rho \nabla\cdot\vec{v}\right) = \frac{DU}{Dt} + \frac{p}{\rho}\nabla\cdot v = \dot{Q}$$

$$c_v = \frac{dU}{dT} \quad \Rightarrow \quad \boxed{c_v \frac{DT}{Dt} + \frac{p}{\rho}\nabla\cdot\vec{v} = \dot{Q}}$$

- A choice for $\dot{Q}$ could be a heat ~~diffusion~~ flux, $\dot{Q} = -\nabla\cdot\vec{q}$

where $q = -K\nabla T$ downgradient heat flux: Fourier's law

$$\Rightarrow \quad \boxed{c_v \frac{DT}{Dt} = -\frac{p}{\rho}\nabla\cdot\vec{v} + \nabla\cdot(K\nabla T)}.$$

Isentropic Evolution

- Use $p = \rho R T$ in the thermodynamic equation:

$$c_v \frac{DT}{Dt} - \frac{p}{\rho^2}\frac{D\rho}{Dt} = K\nabla^2 T$$

$$c_v\left(\frac{1}{\rho R}\frac{Dp}{Dt} - \frac{p}{\rho R^2}\frac{D\rho}{Dt}\right) - \frac{p}{\rho^2}\frac{D\rho}{Dt} = \frac{K}{R}\nabla^2 T$$

$$c_v\left(\frac{1}{\rho R}\frac{Dp}{Dt} - \frac{p\,(c_v+R)}{\rho^2 R c_v}\frac{D\rho}{Dt}\right) = \frac{K}{R}\nabla^2\left(\frac{p}{\rho}\right)$$

- Introduce $\gamma = \frac{c_v+R}{c_v} = \frac{c_p}{c_v}$

$$\frac{c_v}{\rho R}\left(\frac{Dp}{Dt} - \frac{\gamma p}{\rho}\frac{D\rho}{Dt}\right) = \frac{K}{R}\nabla^2\left(\frac{p}{\rho}\right)$$

$$\Rightarrow \frac{1}{\gamma-1}\left(\frac{Dp}{Dt} - \frac{\gamma p}{\rho}\frac{D\rho}{Dt}\right) = \rho\frac{K}{R}\nabla^2\left(\frac{p}{\rho}\right)$$

- If there is no heating, then

$$\frac{Dp}{Dt} - \frac{\gamma p}{\rho}\frac{D\rho}{Dt} = 0$$

$$\Leftrightarrow \frac{D}{Dt}\left(\frac{p}{\rho^\gamma}\right) = \frac{1}{\rho^\gamma}\left(\frac{Dp}{Dt} - \frac{\gamma p}{\rho}\frac{D\rho}{Dt}\right) = 0.$$

$$\boxed{\frac{D}{Dt}\left(\frac{p}{\rho^\gamma}\right) = 0}$$ Isentropic Evolution

- If $\frac{p}{\rho^\gamma}$ is initially uniform in space, it will stay uniform:

$$\frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^\gamma$$ Homentropic evolution.

Gas Dynamics and the speed of sound

- Equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \qquad \text{Continuity}$$

$$\rho \left(\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} \right) = -\nabla p \qquad \text{Momentum}$$

$$\frac{D}{Dt}\left(\frac{p}{\rho^\gamma} \right) = 0 \qquad \text{Thermodynamic (Isentropic evolution).}$$

- Linearize about a state of rest:

$\rho = \rho_0 + \rho'$

$v = v'$

$p = p_0 + p'$

$$\leadsto \quad \frac{\partial \rho'}{\partial t} + \rho_0 \nabla \cdot v' = 0 \qquad (1)$$

$$\rho_0 \frac{\partial \vec{v}'}{\partial t} = -\nabla p' \qquad (2)$$

Thermodynamic:

$$\frac{D}{Dt}\left(\frac{p}{\rho^\gamma} \right) = \frac{\partial}{\partial t} (p \rho^{-\gamma})$$

$$= \frac{\partial}{\partial t} \left[p_0 \rho_0^{-\gamma} \left(1 + \frac{p'}{p_0}\right)\left(1 + \frac{\rho'}{\rho_0}\right)^{-\gamma} \right]$$

$$\approx \frac{\partial}{\partial t} \left[p_0 \rho_0^{-\gamma} \left(1 + \frac{p'}{p_0}\right)\left(1 - \frac{\gamma \rho'}{\rho_0}\right) \right]$$

$$\approx \frac{\partial}{\partial t} \left[p_0 \rho_0^{-\gamma} \left(\frac{p'}{p_0} - \frac{\gamma \rho'}{\rho_0} \right) \right] = 0$$

$$\Longrightarrow \quad p' = \frac{\gamma p_0}{\rho_0} \rho' \qquad (3)$$

- Define $c_s^2 = \dfrac{\gamma p_0}{\rho_0}$ so $p' = c_s^2 \rho'$ $\Longrightarrow$ $c_s^2 = \left(\dfrac{\partial p}{\partial \rho} \right)_T$.

Then (1), (2) imply

$$\boxed{\frac{\partial^2 \rho'}{\partial t^2} = c_s^2 \nabla^2 \rho'} \qquad \text{pressure waves}$$

9 The Navier-Stokes Equation

Derivation

- Start with momentum equation

$$\rho\left(\frac{\partial \vec{v}}{\partial t} + \vec{v}\cdot\nabla\vec{v}\right) = \rho\vec{g} + \nabla\cdot\overleftrightarrow{\Pi}$$

- Decompose stress tensor into pressure and shear components

$$\rho\left(\frac{\partial \vec{v}}{\partial t} + \vec{v}\cdot\nabla\vec{v}\right) = \rho\vec{g} - \nabla p + \nabla\cdot\overleftrightarrow{\tau}$$

- For Newtonian Fluids, $\overleftrightarrow{\tau} = \mu\nabla\vec{v}$

$$\rho\frac{D\vec{v}}{Dt} = \rho\vec{g} - \nabla p + \nabla\cdot(\mu\nabla\vec{v})$$

- Uniform viscosity:

$$\rho\frac{D\vec{v}}{Dt} = \rho\vec{g} - \nabla p + \mu\nabla^2\vec{v}$$

- Define kinematic viscosity $\nu = \frac{\mu}{\rho}$

$$\boxed{\frac{\partial \vec{v}}{\partial t} + \vec{v}\cdot\nabla\vec{v} = -\rho\vec{g} - \nabla p + \nu\nabla^2\vec{v}}$$ Navier-Stokes equation.

- Helpful to remember $\frac{1}{\rho_0}\nabla\cdot\overleftrightarrow{\tau} = \nu\nabla^2\vec{v}$.

so for instance $\nu\frac{\partial^2 u}{\partial z^2} = \frac{1}{\rho_0}\frac{\partial \tau^{(x)}}{\partial z}$.

Boundary conditions

~~1) No-slip~~

- NS involves second-order derivatives so we will need two boundary conditions for $\vec{v}$ instead of just one.
  We will continue to assume no normal flow $n\cdot(\vec{v}-\vec{v}_b)$ for the 1st BC.
  Choices for the 2nd BC:
  1) No slip: with no normal flow this becomes

Non-dimensionalized Navier-Stokes

$$\frac{L}{TU}\frac{\partial \vec{v}}{\partial t} + \vec{v}\cdot\nabla\vec{v} = \frac{gL}{U^2}\hat{g} - \frac{}{U^2}\nabla p + \frac{\nu}{UL}\nabla^2\vec{v}$$

$$\rightsquigarrow \boxed{St\,\frac{\partial \vec{v}}{\partial t} + \vec{v}\cdot\nabla\vec{v} = \frac{1}{Fr}\hat{g} - \nabla p + \frac{1}{Re}\nabla^2\vec{v}}$$

$St = \frac{L}{TU}$: time-varying vs advective terms

$Fr = \frac{U^2}{gL}$: advective vs gravitational forces

$Re = \frac{UL}{\nu}$ advective vs viscous forces

- Flow regimes
  - If $Re \gg 1$, fluid acts like an ideal fluid
    - * Unstable to perturbations → <u>turbulence</u>
  - If $Re \ll 1$, viscous terms dominate over advection
    - * Laminar flow.
  - Even for $Re = \frac{UL}{\nu}$ large, viscous effects can become important in boundary layers $l \ll 1$ near interfaces, where $\frac{Ul}{\nu} \sim 1$.

Vorticity Equation

- Take curl of momentum and use $\nabla^2\vec{v} = \nabla(\nabla\cdot\vec{v}) - \nabla\times(\nabla\times\vec{v})$

$$\frac{D\vec{v}}{Dt} = -\nabla\left(\frac{p}{\rho_0}\right) + \nabla\phi + \nu\nabla^2\vec{v}$$

$$\rightsquigarrow \frac{D\vec{\omega}}{Dt} = \vec{\omega}\cdot\nabla\vec{v} + \nabla\times(\nu\nabla^2\vec{v})$$

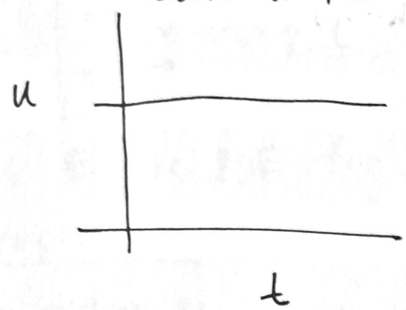
$$= \vec{\omega}\cdot\nabla\vec{v} - \nabla\times\nu(\nabla\times\vec{\omega})$$

$$= \vec{\omega}\cdot\nabla\vec{v} + \nu\nabla^2\vec{\omega}.$$

$$\rightsquigarrow \boxed{\frac{D\vec{\omega}}{Dt} = \vec{\omega}\cdot\nabla\vec{v} + \nu\nabla^2\vec{\omega}}$$

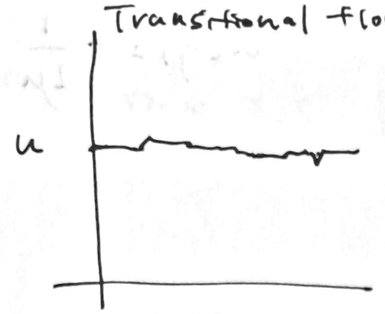
Transition from laminar to turbulent flow

Laminar flow



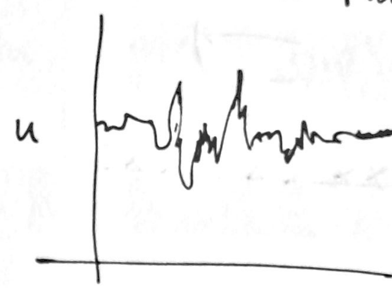
Low Re

Transitional flow



Intermediate Re ~~Transition~~

Turbulent flow



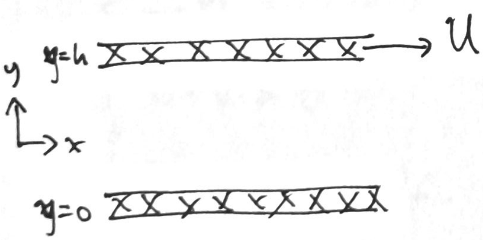
High Re

Exact laminar solutions

- Steady, incompressible NS equations

$$\vec{v}\cdot\nabla\vec{v} = -\frac{1}{\rho_0}\nabla p + \nu\nabla^2\vec{v}$$

- Couette flow: BCs $\vec{u}(x,0)=0$, $\vec{u}(x,h)=0$.



- Want flow between two plates
- Solve for flow $\vec{u}=(u(y), v(y))$
- Incompressibility:

$$u_x + v_y = v_y = 0$$

$$\Rightarrow v = \text{const} = 0 \quad (\text{since } v(0)=0).$$

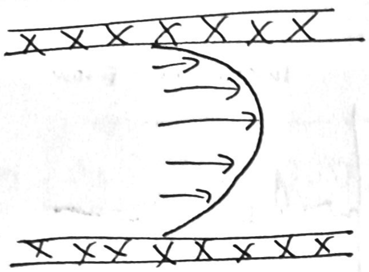
$$\vec{u}\cdot\nabla\vec{u} = -\frac{1}{\rho_0}\nabla p + \nu\nabla^2\vec{u}$$

$\Rightarrow$

Since $p=p(x)$ and $u=u(y)$

these are both constant:

• poiseuille flow:



• Pressure driven flow between two plates:

$$u(y) = -\frac{1}{2\mu}\frac{dp}{dx}\, y(h-y)$$

11 Stokes flow.

Governing equations

$$\begin{cases} \nabla^2 \vec{v} = \nabla p \\ \nabla \cdot \vec{v} = 0 \end{cases} \quad \text{or} \quad \begin{cases} \nabla \times \omega = -\nabla p \\ \nabla \cdot v = 0 \end{cases} \quad \text{via} \quad \nabla^2 \vec{v} = \nabla(\nabla \cdot \vec{v}) - \nabla \times (\nabla \times \vec{v})$$

Elliptic PDE for $\vec{v}$.

Properties

- Stokes flow is unique.
- Stokes flow is time-reversible: since it is uniquely determined by the boundary condition, bringing it back to the original location results in the same flow.

Uniqueness

- Suppose $\vec{v}_1$ and $\vec{v}_2$ solve

$$\begin{cases} \nabla^2 \vec{v}_1 = \nabla p_1 \\ \nabla \cdot \vec{v}_1 = 0 \end{cases} \quad \begin{cases} \nabla^2 \vec{v}_2 = \nabla p_2 \\ \nabla \cdot \vec{v}_2 = \nabla p_2 \end{cases} \quad \vec{x} \in \Omega, \quad \vec{v}_1 = \vec{v}_2 = \vec{v}_b \text{ on } x \in \partial\Omega$$

Then $\vec{v} = \vec{v}_1 - \vec{v}_2$ solves

$$\begin{cases} \nabla^2 \vec{v} = \nabla p \\ \nabla \cdot \vec{v} = 0 \end{cases} \quad x \in \Omega \qquad p = p_1 - p_2$$

$$\vec{v} = 0, \quad x \in \partial\Omega$$

- Rewrite using vector Laplacian

$$\nabla^2 \vec{v} = \nabla(\nabla \cdot v) - \nabla \times \nabla \times \vec{v}$$

$$\Rightarrow \begin{cases} \nabla \times \nabla \times \vec{v} = -\nabla p \\ \nabla \cdot \vec{v} = 0 \end{cases} \quad x \in \Omega$$

$$\Rightarrow \int_\Omega \vec{v} \cdot \underbrace{(\nabla \times \vec{\omega} + \nabla p)}_{=0} \, dV = \dots = \int_\Omega |\nabla \times \vec{v}|^2 \, dV = 0$$

$$\Rightarrow \boxed{\nabla \times \vec{v} = 0 \text{ everywhere in } \Omega}$$

- Since $\nabla \times \vec{v} = 0$, can define a potential $\vec{v} = \nabla \phi$

$\Rightarrow$ ~~$\nabla^2\phi$~~ $\nabla \cdot \vec{v} = \nabla^2 \phi = 0$

ϕ satisfies laplace with $\nabla \phi = 0$ on $x \in \partial \Omega$

$\Rightarrow \phi = \text{const} \Rightarrow \boxed{\vec{v} = 0}$.