Characteristics and Shocks

The transport equation

$$u_t + c u_x = 0$$

If $\frac{dX(t)}{dt} = c$, then $\frac{d}{dt}\left(u(X(t),t)\right) = u_x \frac{dX}{dt} + u_t = 0$

So on characteristics $X(t) = ct + b$, the solution is constant

$\Rightarrow$ If we have an initial profile $u(x,0) = u_0(x)$,

then the solution is given by ~~$u_0(x-ct)$ at time t~~

$$u(x,t) = u_0(x-ct) \quad \text{at time } t.$$
$$= u(x-ct,0) = u_0(x-ct)$$

Nonuniform transport

$$\frac{\partial u}{\partial t} + c(x,t)\frac{\partial u}{\partial x} = f(x,t)$$

If $\frac{dX(t)}{dt} = c(x,t)$, then $\frac{d}{dt}\left\{u(X(t),t)\right\} = f(x,t)$

$\Rightarrow$ Solve ODE for $X(t)$,

solve ODE for $u(X(t),t)$.

Burger's equation

$$u_t + u u_x = 0$$

- Again can apply characteristics: consider curve $X(t)$ such that

$$\frac{dX}{dt} = u(x,t). \quad \text{Then}$$

$$\frac{d}{dt}\left\{u(X(t),t)\right\} = u_t + u_x \cdot \frac{dX}{dt} = u_t + u u_x = 0.$$

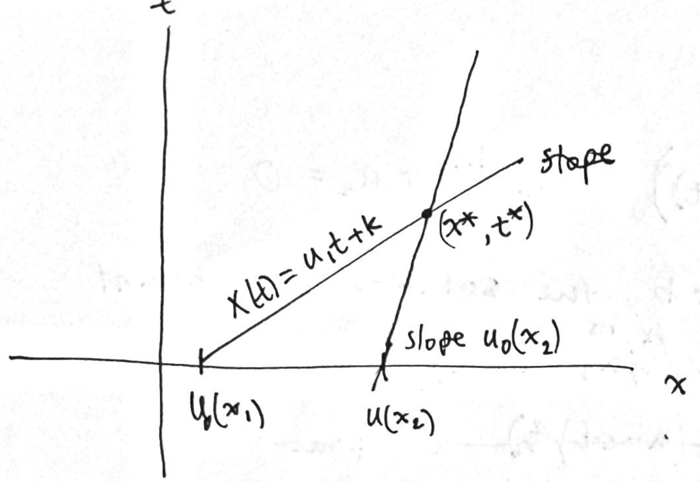
- Because $u(X(t),t) = \text{const}$ along characteristics, $\frac{dX}{dt} = u(x,t) = u(x,0)$

$\Rightarrow X(t) = ut + k$.

- If $u(x,0) = u_0(x)$, then $u(x,t) = u_0(x-ut)$ implicitly defines a solution.

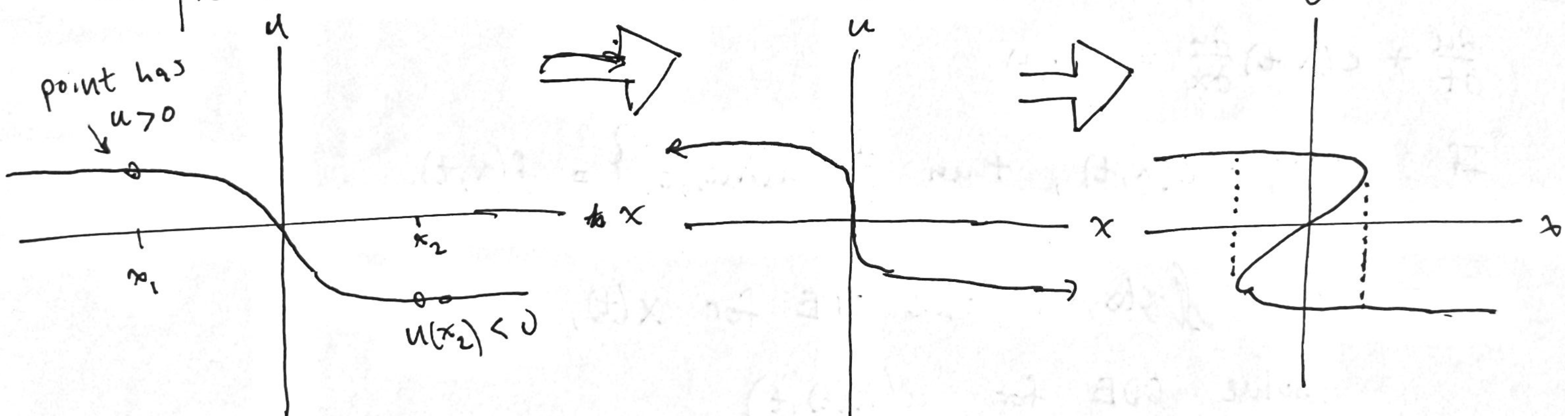
Shocks

- Draw characteristics for initial values of u in x,t plane:



- u is constant along characteristics until the model breaks down at (x^*, t^*).

- At (x^*, t^*), nonparallel characteristics cross each other
- Example with initial data $u_0(x) = -\tan^{-1}(x)$



$t = 0$ $\quad$ $t = t^*$ $\quad$ $t > t^*$

- Breakdown of ~~finite~~ unique solution in finite time t^*
- At $t > t^*$, some given points x admit 3 values of u.
- Need to make a decision on how to define the solution post-shock, probably based on the physical properties of the system.

Traffic Flow

Variables

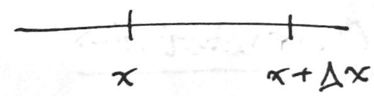
- $\rho(x,t)$ density of cars
- $u(x,t)$ mean velocity of cars
- $Q(x,t)$ flux of cars

$$Q = \rho u$$

Conservation equations and weak form

- ~~Flux~~ Change of cars between $(x, x+\Delta x)$, during $(t, t+\Delta t)$:

$$= \cancel{\int\int \rho(x,t_2) - \rho(x,t_1)}$$



$$= \int_x^{x+\Delta x} \rho(\tilde{x}, t+\Delta t) - \rho(\tilde{x}, t)\, d\tilde{x}$$

- ~~An~~ This is also equal to the net flux in over this time span

$$= \int_t^{t+\Delta t} Q(x,\tilde{t}) - Q(x+\Delta x, \tilde{t})\, d\tilde{t}$$

- Use FTC on each of these:

$$\Rightarrow \int_x^{x+\Delta x}\int_t^{t+\Delta t} \frac{\partial \rho}{\partial t}\, dt\, dx = \int_t^{t+\Delta t}\int_x^{x+\Delta x} -\frac{\partial Q}{\partial x}\, dx\, dt$$

$$\Rightarrow \boxed{\int_x^{x+\Delta x}\int_t^{t+\Delta t} \rho_t + Q_x\, dt\, dx = 0}$$

weak formulation of car conservation.

smooth derivatives $\Rightarrow$ $\boxed{\rho_t + Q_x = 0}$

Conservation law

$\rho_t + Q_x = 0$ 2 unknowns, on equation

- Make closure by supposing $Q = Q(\rho)$

$\Rightarrow \rho_t + Q'(\rho)\,\rho_x = 0$ $\boxed{\text{Conservation equation } \quad u_t + A(u)\,u_x = 0}$

- Functional form of $Q(\rho)$:

$Q = \rho u$, and $u = u(\rho)$ should be decreasing.

Take $Q(\rho) = \rho(1-\rho)$ as a simple example.

Characteristics

- Turn PDE into ODE by restricting ourselves to characteristic curves.
- Note that if $X = X(t)$ s.t. $\frac{dX}{dt} = Q'(\rho)$, then

$$\frac{d}{dt}\left(\rho(X(t),t)\right) = \rho_t + \frac{dX}{dt}\rho_x = \rho_t + Q'(\rho)\,\rho_x = 0.$$

- In Conclusion: on ~~curves~~ (lines) $X(t) = Q'(\rho)t + \cancel{x_0}\,\xi$, the density does not change:

$$\rho(X(t),t) = \rho\left(Q'(\rho)t + x_0, t\right) = \text{const}.$$

- Density becomes a function of the characteristic variable

$$x_0 = \xi = X(t) - Q'(\rho)t = x_0$$

$$\rho(X(t),t) = \rho_0(\xi) = \rho_0\left(x - Q'(\rho(x,t))t\right)$$

· Recap:

- On the straight lines $X(t) = Q'(\rho)t + x_0$, the density does not change: $\rho(X(t),t) = \text{const} = \rho(x_0,t) =: \rho_0(x)$

- Density $\rho(x,t)$ therefore only depends on the characteristic variable $x_0 = X(t) - Q'(\rho)t$:

$$\rho(x,t) = \rho_0(\xi) = \rho_0(x - Q'(\rho(x,t))\,t)$$

- Implicitly defines ρ. Only well defined until $\rho_x(x,t) = \infty$:

$$\rho_x(x,t) = \rho_0'(x - Q'(\rho(x,t))t)\left[1 - Q''(\rho(x,t))\,\rho_x(x,t)\,t\right]$$

$$\Rightarrow \quad \rho_x = \frac{\rho_0'(x_0)}{1 + \rho_0'(x_0)\,Q''(\rho_0(x_0))\,t}$$

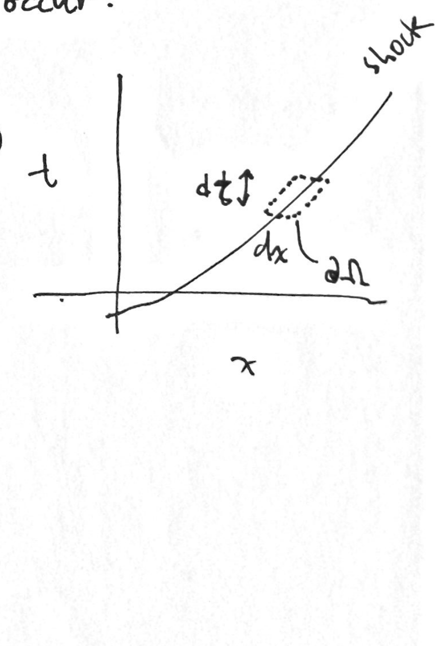
$$\Rightarrow \text{ blow up at } t = \frac{-1}{\rho_0'(x_0)\,Q''(\rho_0(x_0))}$$

Shocks / Rankine-Hugoniot condition

· What happens post shock? Where does the shock occur?

· Weak form of conservation:

$$\int_\Omega \rho_t + Q_x \, dx\, dt = \oint_{\partial\Omega} \rho\, dx - Q\, dt = 0$$



· Shock speed:

$$c = \frac{dx}{dt} = \frac{Q(\rho^+) - Q(\rho^-)}{\rho^+ - \rho^-}$$

Conservation laws

Definition: Conservation law

- $u = u(x,t) \in \mathbb{R}^n$
- $u_t + F(u)_x = 0$
- Weak form: $\oint_{\partial\Omega} u\,dx - F\,dt$
- If F is differentiable:

$$u_t + A(u)\,u_x \quad \text{where} \quad A_{ij} = \left(\frac{\partial F_i}{\partial u_j}\right)$$

Examples

- Shallow water:

$$\cancel{h_t + (hu)_x = 0}$$

$$\cancel{(hu)_t + \left(hu^2 + g\frac{h^2}{2}\right)_x = 0}$$

$$h_t + u h_x + h u_x = 0$$

$$u_t + u u_x + g h_x = 0$$

Let $u = \begin{bmatrix} h \\ u \end{bmatrix}$, $A = \begin{bmatrix} u & h \\ g & u \end{bmatrix}$

$$\rightsquigarrow u_t + A u_x = 0.$$

- Gas dynamics:

$$\rho_t + (\rho u)_x = 0$$

$$(\rho u)_t + (\rho u^2 + p)_x = 0$$

$$\left(e + \frac{\rho u^2}{2}\right)_t + \left(eu + \rho\frac{u^3}{2} + pu\right)_x = 0$$

can be written in conservation form.

Riemann Invariants

If we have a full system of real eigenvalues, then

$$l_j^T A = \lambda_j l_j^T \quad \text{left eigenvectors}$$

$$\Rightarrow \quad U_t + A U_x = 0$$

$$\Rightarrow \quad l_j^T U_t + \lambda_j l_j^T U_x = 0$$

Define the Riemann invariants $R_j = l_j^T U$

$$\Rightarrow \quad \frac{\partial}{\partial t} R_j + \lambda_j \frac{\partial}{\partial x} R_j = 0$$

so characteristics $\frac{dX_j}{dt} = \lambda_j$ ~~simply~~ yield curves where the Riemann Invariants are constant.

- Example: shallow water system.

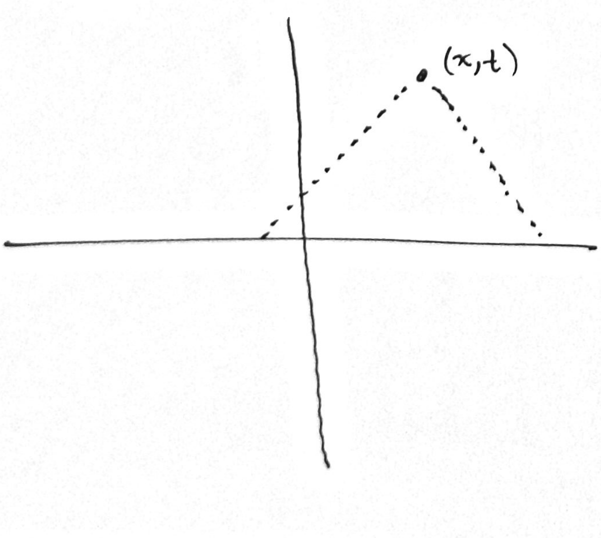
$$U_t + A U_x = 0 \qquad A = \begin{bmatrix} u & h \\ g & u \end{bmatrix} \qquad U = \begin{bmatrix} h \\ u \end{bmatrix}$$

$$\lambda_1 = u + \sqrt{gh} \qquad l_1^T = \left(\sqrt{\tfrac{g}{h}}, 1\right) \Rightarrow R_1 = u + 2\sqrt{gh}$$

$$\lambda_2 = u - \sqrt{gh} \qquad l_2^T = \left(-\sqrt{\tfrac{g}{h}}, 1\right) \Rightarrow R_2 = u - \sqrt{gh}$$

- Suppose we have initial data $h_0(x) = \begin{cases} 1, & |x| < 1 \\ 0 & \text{otherwise} \end{cases}$; $u_0(x) = 0$

To find solution at (x,t), trace back ~~solution~~ characteristics to origin:



~~Simple waves~~

Simple waves

$$U_t + A(U)U_x = 0$$

Suppose solution of the form $U(\theta(x,t))$

Then $\frac{\partial}{\partial t}(U) = U'(\theta)\,\theta_t$, $\frac{\partial}{\partial x}(U) = U'(\theta)\,\theta_x$

$$\rightsquigarrow \left(A(U) + \frac{\theta_t}{\theta_x} I\right) U'(\theta) = 0$$

works if $\lambda = \frac{-\theta_t}{\theta_x}$ is an eigenvalue of $A(U)$ with eigenvector $U'(\theta)$

Dispersive waves

- One of the most important ideas in the study of differential equations is that derivatives of exponential functions are exponential functions.
- We can exploit this by making a guess $u(x,t) = e^{i(kx - \omega t)}$ and figuring out what ω has to be later: $\omega = \Omega(k)$
- Example: linearized shallow water equations:

$$u_t - fv = -g\eta_x \qquad (h = H + \eta).$$
$$v_t + fu = -g\eta_y$$
$$\eta_t + H(u_x + v_y) = 0$$

  - These equations can be reduced to a single equation for η:

$$\eta_{tt} + f_0^2\eta - gH\Delta\eta = 0$$

  - Making an ansatz $\eta = e^{i(k\cdot x - \omega t)}$ yields

$$\eta_{tt} = (-i\omega)^2\eta = -\omega^2\eta$$
$$\Delta\eta = -(k^2 + \ell^2)$$

$\Rightarrow$ ~~[illegible] equation~~

$$-\omega^2\eta + f_0^2\eta + gH(k^2+\ell^2)\eta = 0$$

so the PDE is solved if

$$\omega^2 = f_0^2 + gH(k^2+\ell^2).$$

- Then, we can find the general solution by superposition of the plane wave solutions:

$$u(x,t) = \int \hat{u}(k)\, e^{i(k\cdot x - \omega t)}\, dk.$$

  - At $t = 0$, we recover

$$u_0(x) = \int \hat{u}(k)\, e^{i(k\cdot x)}\, dk$$

  - So $\hat{u}(k)$ is given by the Inverse Fourier transform of initial data:

$$\hat{u}(k) = \frac{1}{2\pi}\int u_0(x)\, e^{-i(k\cdot x)}\, dx.$$

Modulated waves

- Consider more general waveform:

$$u(x,t) = A(x,t)\, e^{i\theta(x,t)}$$

$A(x,t)$ slowly varying

$\theta(x,t)$ can be expanded:

$$\theta(x,t) = \theta(x_0,t_0) + \nabla\theta(x_0,t_0)\cdot(x-x_0) + (t-t_0)\frac{\partial\theta}{\partial t}(x_0,t_0) + \mathcal{O}(\Delta x^2)$$

$$= \theta_0 + \nabla\theta\cdot\Delta x + \frac{\partial\theta}{\partial t}\Delta t .$$

- Define wavenumber and frequency:

$$k(x,t) = \nabla_x\theta \qquad \omega(x,t) = -\theta_t$$

Then $u(x,t) = A(x,t)\, e^{i\theta(x,t)}$

$$\approx A(x,t)\, e^{i(k\cdot\Delta x - \omega\Delta t + \theta_0)}$$

We can solve for $\omega = \Omega(k, x, t)$ that solves the PDE.

Group velocity

- Conservation of waves:

$$\frac{\partial}{\partial t}\nabla_x\theta - \nabla_x\frac{\partial}{\partial t}\theta = 0 \implies k_t + \nabla_x\omega = 0 .$$

- $\omega = \Omega(k(x,t), x, t) \implies$

$$\nabla_x\omega = \frac{\partial\Omega}{\partial x_i} = \frac{\partial\Omega}{\partial x_i} + \frac{\partial\Omega}{\partial k_j}\frac{\partial k_j}{\partial x_i} = \frac{\partial\Omega}{\partial x_i} + \frac{\partial\Omega}{\partial k_j}\frac{\partial k_i}{\partial x_j}$$

$$= \nabla_x\Omega + \vec{c}_g\cdot\nabla_x k$$

where $\vec{c}_g = \left(\frac{\partial\omega}{\partial k_j}\right)_{j=1}^{n} = \nabla_k\Omega$ is the group velocity

- $k_t + \nabla_x\omega = 0 \implies$

$$k_t + c_g\cdot\nabla_x k = -\nabla_x\Omega$$

- So the group velocity is the characteristic line for the wave:

along $\frac{dx}{dt} = c_g, \quad \dot{k} = -\nabla_x\Omega .$

The Wave Equation

$$u_{tt} - c^2 \nabla^2 u = 0 \qquad \text{Hyperbolic PDE.}$$

D'Alembert's solution

- Factor out wave operator:

$$\Box u = (\partial_t^2 - c^2 \partial_x^2) u = (\partial_t - c\,\partial_x)(\partial_t + c\,\partial_x) u$$

- Note that $(\partial_t + c\,\partial_x) u = 0 \Rightarrow \Box u = 0$, similar to $(\partial_t - c\,\partial_x) u = 0$.

- Theorem: Any solution to the (1D) wave equation can be written as $u(t,x) = p(x-ct) + q(x+ct)$

- Initial value problems:

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u(0,x) = f(x) \\ u_t(0,x) = g(x) \end{cases}$$

$$\Rightarrow f(x) = p(x) + q(x) \rightsquigarrow f'(x) = p'(x) + q'(x)$$

$$g(x) = -c\,p'(x) + c\,q'(x)$$

$$\rightsquigarrow 2p'(x) = f'(x) - \frac{1}{c} g(x)$$

$$\Rightarrow p(x) = \frac{1}{2} f(x) - \frac{1}{2c} \int_0^x g(z)\,dz + \text{const}$$

$$\Rightarrow q(x) = f(x) - p(x) = \frac{1}{2} f(x) + \frac{1}{2c} \int_0^x g(z)\,dz - \text{const}$$

$$\Rightarrow \boxed{u(t,x) = \frac{f(x-ct) + f(x+ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s)\,ds}$$

Duhamel's principle

- Forced wave equation
$$u_{tt} + c^2 u_{xx} = F(x,t)$$
- We only need to consider homogeneous BCs due to superposition:
$$\begin{cases} \Box u = F(x,t) \\ u(x,0) = 0 \\ u_t(x,0) = 0 \end{cases}$$
- Idea: $(u_t)_t \approx F(x,t)$ F acts to "impulse" u_t. Try superimposing solutions to IVPs $\begin{cases} u(x,t_0)=0 \\ u_t(x,t_0) = F(x,t_0) \end{cases}$
- Denote $\mathcal{U}(x,t;s)$ as a solution for $t \geq s$ to
$$\begin{cases} \Box \mathcal{U} = 0 \\ \mathcal{U}(x,s;s) = 0 \\ \mathcal{U}_t(x,s;s) = F(x,s) \end{cases}$$

Then the solution is
$$u(x,t) = \int_0^t \mathcal{U}(x,t;s)\,ds .$$

Proof:

$$\Box u(x,t) = \Box \int_0^t \mathcal{U}(x,t;s)\,ds$$
$$= \partial_{tt}\int_0^t \mathcal{U}(x,t;s)\,ds + \int_0^t -c^2\,\partial_{xx}\mathcal{U}(x,t;s)\,ds$$
$$= \partial_t\Big[\int_0^t \mathcal{U}_t(x,t;s)\,ds + \underbrace{\mathcal{U}(x,t;t)}_{=0}\Big] + \int_0^t -c^2\,\mathcal{U}_{xx}(x,t;s)\,ds$$
$$= \int_0^t \mathcal{U}_{tt}(x,t;s)\,ds + \mathcal{U}_t(x,t;t) - \int_0^t c^2\,\mathcal{U}_{xx}(x,t;s)\,ds$$
$$= \int_0^t \Box\mathcal{U}\,ds + \mathcal{U}_t(x,t;t) = F(x,t), \text{ as needed.}$$

Method of Spherical Means

Goal: solution to wave equation in higher dimensions.

- Introduce spherical mean:

For $u(x,t)$ solving

$$\begin{cases} u_{tt} - c^2 \Delta u = 0 \\ u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}$$

Define $\displaystyle U(x,t,r) = \int_{B(x,r)} u(s,t)\, dS = \int_{B(x,1)} u(x+r\psi,t)\, dS$ ← unit vector

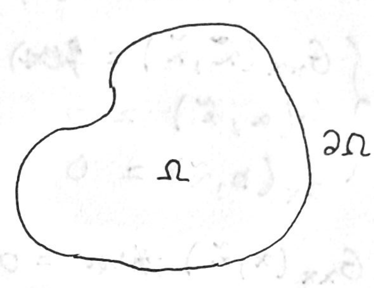
Then $u(x,t) = \lim_{r\to 0} U(x,t,r)$ (average over infinitely small sphere).

- Note that U solves the wave equation for spherically symmetric waves:

$$U_{tt} - c^2\left(U_{rr} + \frac{n-1}{r} U_r\right) = 0 .$$

Laplace and Poisson Equations

$$\begin{cases} \Delta u = f(x); & x \in \Omega \\ u\big|_{\partial\Omega} = g(x) \quad \text{or} \quad u(x)\cdot n = g(x) & \text{on } x \in \partial\Omega \end{cases}$$



- Laplace: $f(x) = 0$
- Elliptic PDE.

1D Poisson Equation

$$\begin{cases} u''(x) = f(x) & x \in [a,b] \\ u(a) = \alpha \\ u'(b) = \beta \end{cases}$$

for example,

- Superposition principle: decompose into $u = v + w$ where

$$\begin{cases} v''(x) = 0 \\ v(a) = \alpha \\ v'(b) = \beta \end{cases} \qquad \begin{cases} w''(x) = f(x) \\ w(a) = 0 \\ w'(b) = 0 \end{cases}$$

Then $u''(x) = f(x)$, $u(a) = \alpha$, $u(b) = \beta$.

v is easy to solve: must be a linear equation on $[a,b]$.

- Green's function

Homogeneous Poisson:

$$\begin{cases} u''(x) = f(x) \\ u(a) = 0 \\ u'(b) = 0 \end{cases} \qquad (*)$$

Take Green's function satisfying

$$\begin{cases} G_{xx}(x,\tilde{x}) = \delta(x-\tilde{x}) \\ G(a,\tilde{x}) = 0 \\ G_x(b,\tilde{x}) = 0 \end{cases}$$

Then $u(x) = \int_a^b G(x,\tilde{x}) f(\tilde{x})\, d\tilde{x}$ solves $(*)$

Need to find $G(x,\tilde{x})$ solving

$$\begin{cases} G_{xx}(x,\tilde{x}) = \cancel{f(x)}\,\delta(x-\tilde{x}) \\ G(a,\tilde{x}) = 0 \\ G_x(b,\tilde{x}) = 0 \end{cases}$$

* $G_{xx}(x,\tilde{x}) = 0$ for $x \neq \tilde{x}$ implies, with the boundary conds

$$G(x,\tilde{x}) = \begin{cases} c_1(x-a), & x < \tilde{x} \\ \cancel{}\ c_2 & x > \tilde{x} \end{cases}$$

$bx +$

* G is continuous at $x=\tilde{x}$:

$$\Rightarrow \quad c_1(x-a) = bx + c_2$$

* Jump condition:

$$G_x(x^+,\tilde{x}) - G_x(x^-,\tilde{x}) = 1$$

~~together~~

- General principles:

~~[illegible]~~

* $G_{xx}(x,\tilde{x}) = \delta(x-\tilde{x})$ implies that G is linear for $x \neq \tilde{x}$.

* G satisfies the boundary conditions.

* $G = \iint G_{xx}\,dx\,dx$ implies G is continuous at $x=\tilde{x}$.

* $G_x = \int \delta(x-\tilde{x})\,dx \Rightarrow G_x(x^+,\tilde{x}) - G_x(x^-,\tilde{x}) = 1$.

Poisson's Equation in Higher dimensions

$$\Delta u = f(x) \quad x \in \Omega$$
$$u(x) = u_b(x) \quad x \in \partial\Omega$$

consider $u = u_1 + u_2$

$$\begin{cases} \Delta u_1 = 0 & x \in \Omega \\ u_1(x) = u_b(x) & x \in \partial\Omega \end{cases} \qquad \begin{cases} \Delta u_2 = f(x) & x \in \Omega \\ u_2(x) = 0 & x \in \partial\Omega \end{cases}$$

For u_2: consider Green's functions

$$\begin{cases} \Delta G(x; \tilde{x}) = \delta(x - \tilde{x}) \\ G(x, \tilde{x}) = 0 \end{cases}$$

$$u_2(x) = \int_\Omega G(x; \tilde{x}) f(\tilde{x}) \, d\tilde{x}.$$

For u_1: Invoke Maxwell's Reciprocity principle

$$\int_\Omega (u \Delta v - v \Delta u) \, dx = \int_{\partial\Omega} (u \nabla v - v \nabla u) \cdot n \, dx$$

$$\Rightarrow \int_\Omega u_1 \Delta G - G \Delta u_1 \, dx = \int_{\partial\Omega} (u_1 \nabla G - G \nabla u_1) \cdot n \, dx$$

$$\rightsquigarrow u_1(\tilde{x}) = \int_{\partial\Omega} u_b(x) \nabla_x G(x, \tilde{x}) \cdot \vec{n} \, d\vec{x}$$

$$\Rightarrow u_1(x) = \int_{\partial\Omega} u_b(\tilde{x}) \nabla_{\tilde{x}} G(x, \tilde{x}) \cdot n \, d\vec{x}$$

Fundamental Solutions to Laplace's equation

- For general domains Ω, no closed form for G.
- For symmetric domains, things are better.
- For $\mathbb{R}^n$:

$$\Delta F = \delta(x)$$

- Invariance under rotation:

$$\left(r^{n-1} F'(r)\right)' = 0, \quad r > 0$$

$$\Rightarrow \boxed{\begin{array}{l} n=2: \quad F = \frac{-1}{2\pi} \log r = \frac{-1}{2\pi} \log \|x\| \\ n=3: \quad F = \frac{1}{4\pi r} = \frac{1}{4\pi} \frac{1}{\|x\|} \end{array}}$$

$$\boxed{n \geq 3: \quad F = \frac{C_n}{r^{n-2}}}$$

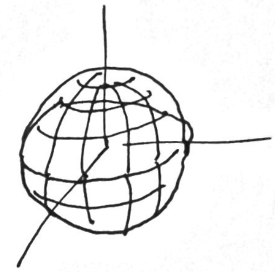
Method of Images:

- For domains like $x_1 > 0$, $x_1, x_2 > 0$, superimpose equivalent sinks to satisfy boundary conditions $G(x_1, \tilde{x}) = 0$

Mean-value properties of Laplace's equation

- Laplacian smoothens fields, e.g. like diffusion
- we shouldn't expect irregularities / local optima within the domain.
- Implications:
  1) Maximum principle: harmonic functions on connected domains cannot attain their maximum on the interior (unless it is constant).
  2) Dirichlet Poisson problem has at most one solution.
  3) Liouville's theorem: Harmonic functions in $\mathbb{R}^n$ bounded neither above or below is a constant.

Mean-Value property of harmonic functions



$$\begin{cases} \Delta u = 0 & x \in \Omega \\ u(x) = u_b(x) & x \in \partial\Omega \end{cases}$$

- $u(x)$ equals its average over an enclosed ball:

- Define

$$\bar{u}(x,r) = \frac{1}{A_r}\int_{\partial B_1(x)} u(x + r\vec{n}(s))\, ds \qquad A_r = \text{Area} = 1 \text{ WLOG (normalising } ds)$$

Then

$$\frac{\partial \bar{u}}{\partial r} = \int_{\partial B_1(x)} \frac{\partial u}{\partial n}(x + r\vec{n}(s))\, ds = \int_{\partial B_1} \nabla u \, ds = \int_{B_1} \nabla^2 u \, dV = 0$$

- Since $u = \lim_{r \to 0} \bar{u}(x,r)$,

$$u(x) = \bar{u}(x,r) \text{ for all } r > 0.$$

The Heat Equation

$$u_t = K \Delta u$$

Parabolic PDE.

Physical origin:

- Heat flows from warmer to cooler areas
- Flux is down-gradient:

$$Q(x,t) = -K(x)\nabla x \quad = -K \nabla u \text{ for uniform material.}$$

- conservation law:

$$\frac{\partial u}{\partial t} + \nabla \cdot Q = 0$$

$$\Rightarrow u_t - K\Delta u = 0.$$

Fundamental solution

- Superposition principle. To solve

$$\begin{cases} u_t = \Delta u & x \in \Omega \\ u(x,0) = u_0(x) \end{cases} \quad (*)$$

Note that if $u_0(x) = a_1 v_0(x) + a_2 w_0(x)$

where $v(x,t)$ and $w(x,t)$ solve

$$\begin{cases} v_t = \Delta v \\ v(x,0) = v_0(x) \end{cases} \qquad \begin{cases} w_t = \Delta w \\ w(x,0) = w_0(x) \end{cases}$$

Then $u = a_1 v(x,t) + a_2 w(x,t)$ solves

$$\begin{cases} u_t = \Delta u \\ u(x,0) = u_0(x). \end{cases}$$

Consider $u_0(x) = \int_\Omega u_0(\xi)\delta(x-\xi)\,d\xi$

solving $\begin{cases} G_t = \Delta_x G \\ G(x,0;\xi) = \delta(x-\xi) \end{cases}$

$\rightarrow G(x,t;\xi)$ solves the heat equation with "laser beam" at $x = \xi$ initially.

Then $u(x,t) = \int_\Omega G(x,t;\xi)\, u_0(\xi)\, d\xi$ solves $(*)$

where $u_0(x) = \int_\Omega u_0(\xi)\,\delta(x-\xi)\,d\xi.$

• 1D solution on $\Omega = \mathbb{R}$

Solve

$$\begin{cases} G_t = G_{xx} \\ G(x,0) = \delta(x) \end{cases}$$

- Heat equation is invariant to stretching:

$$x \to \lambda x \qquad t \to \lambda^2 t$$

Need solution

$$G(\lambda x, \lambda^2 t) = G(x,t)$$

$$\lambda = \frac{1}{\sqrt{t}} \Rightarrow G\left(\frac{x}{\sqrt{t}}, 1\right) = G(x,t) =: \Phi(\xi) \qquad \xi = \frac{x}{\sqrt{t}}$$

$$\Rightarrow G_t = -\frac{1}{2}\frac{x}{t^{3/2}}\Phi'(\xi), \qquad G_{xx} = \frac{1}{t}\Phi''(\xi)$$

$$\Rightarrow \Phi'' + \frac{\xi}{2}\Phi' = 0$$

- Solve this ODE:

$$\Phi(\xi) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$$

$$\boxed{G(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t}}$$

Fundamental solution to 1D heat equation.

Example: use Entropy $\phi(u) = -u \log u$

$$\Phi'(t) = \int \phi'(u) u_t \, dx = \int \phi'(u) u_{xx} \, dx$$

$$= -\int \phi''(u) \|\nabla u\|^2 \, dx > 0$$ is increasing function of time.

Diffusion and Brownian Motion

- Particle doing random walk on a grid

$$P_j^n = \mathbb{P}(x = x_j \text{ at time } t = t_n)$$

$$P_j^{n+1} = \frac{1}{2} P_{j-1}^n + \frac{1}{2} P_{j+1}^n$$

- Can rewrite this as a ~~Crank-Nicolson scheme~~: FTCDS scheme:

$$\frac{P_j^{n+1} - P_j^n}{\Delta t} = \frac{1}{2} \frac{\Delta x^2}{\Delta t} \frac{P_{j+1}^n - 2P_j^n + P_{j-1}^n}{\Delta x^2}$$

- Recover Fokker-Planck equation

$$\Rightarrow \boxed{\rho_t = \nu \rho_{xx}}$$

$$\nu = \lim_{\Delta x, \Delta t \to 0} \frac{1}{2} \frac{\Delta x^2}{\Delta t} \qquad \rho(x,t) = \lim_{\Delta x, \Delta t} P(x_j, t_n)$$

Backward Heat Equation

$$\begin{cases} u_t = -\Delta u \\ u(x,0) = u_0(x) \end{cases}$$

- Given homogenized state, hard to recover initial irregularities.

- ND solution to heat equation via Fourier transform

$$\begin{cases} u_t = \Delta u_{\cancel{xx}} \\ u(x,0) = u_0(x) \qquad x \in \mathbb{R}^n \end{cases}$$

- Suppose $u(x,t) = \int \hat{u}(\underline{k})\, e^{i(\underline{k}\cdot\underline{x} - \omega t)}\, d\underline{k}$
- Dispersion relation

$$-i\omega\cancel{t} = -\|k\|^2$$

$$\omega = \frac{\|k\|^2}{i}$$

$$\Rightarrow u(x,t) = \int \hat{u}(\underline{k})\, e^{ik\cdot x - \|k\|^2 t}\, dk$$

- Impose IC:

$$u(x,0) = \int \hat{u}(\underline{k})\, e^{i\underline{k}\cdot\underline{x}}\, d\underline{k}$$

$$\Leftrightarrow \quad \hat{u}(\underline{k}) = \frac{1}{(2\pi)^n}\int u_0(\tilde{x})\, e^{-i\underline{k}\cdot\tilde{x}}\, d\tilde{x}$$

$$\Rightarrow u(x,t) = \frac{1}{(2\pi)^n}\int\left[\int u_0(\tilde{x})\, e^{-ik\cdot\tilde{x}}\, d\tilde{x}\right] e^{ik\cdot x - \|k\|^2 t}\, dk$$

$$= \frac{1}{(2\pi)^n}\int\int u_0(\tilde{x})\, e^{ik\cdot(x-\tilde{x}) - \|k\|^2 t}\, d\tilde{x}\, dk$$

- Exchange order of integration and complete square in k

$$u(x,t) = \frac{1}{(2\pi)^n}\int\int u_0(\tilde{x})\, e^{-\left\|\vec{k}\sqrt{t} + \frac{i(x-\tilde{x})}{2\sqrt{t}}\right\|^2} d\underline{k}\; e^{-\frac{\|x-\tilde{x}\|^2}{4t}}\, \cancel{dk}\, d\tilde{x}$$

$$= \frac{1}{(2\pi)^n}\int \cancel{\int} \left(\frac{\pi}{t}\right)^{\frac{n}{2}} u_0(\tilde{x})\, e^{-\frac{\|x-\tilde{x}\|^2}{4t}}\, d\tilde{x}.$$

$$\boxed{u(x,t) = \frac{1}{(4\pi t)^{n/2}}\int e^{-\left(\frac{\|x-y\|^2}{4t}\right)} u_0(y)\, dy}$$

ND solution to the heat equation.

Duhamel's Principle for the forced heat equation

$$\begin{cases} u_t - \Delta u = f(x,t) \\ u(x,0) = u_0(x) \end{cases} \qquad (*)$$

Introduce $\mathcal{U}(x,t;s)$ satisfying

$$\begin{cases} \mathcal{U}_t = \Delta_{(x)} \mathcal{U} \\ \mathcal{U}(x,s;s) = f(x,s) \end{cases}$$

Then $u = u_h + \int_0^t \mathcal{U}(x,t;s)\,ds$ solves $(*)$

where u_h solves $\begin{cases} \frac{\partial u_h}{\partial t} = \Delta u_h . \\ u_h(x,0) = u_0(x) \end{cases}$

Proof:

$$u_t - \Delta u = \frac{\partial u_h}{\partial t} - \Delta u_h + \frac{\partial}{\partial t}\int_0^t \mathcal{U}(x,t;s)\,ds - \Delta \int_0^t \mathcal{U}(x,t;s)\,ds$$

$$(\text{LIT.}) \quad = \int_0^t \mathcal{U}_t(x,t;s)\,ds + 1 \cdot \mathcal{U}(x;t;t) - \int_0^t \Delta \mathcal{U}(x,t;s)\,ds$$

$$= \mathcal{U}(x,t;t) = f(x,t) . \ \checkmark$$

Maximum Principle

- u does not develop local extrema in the interior
- Uniqueness of solutions: If u_1, u_2 satisfy

$$\begin{cases} u_t = \Delta u + f(x,t), & x \in \Omega \\ u(x,t) = g(x,t) & x \in \partial\Omega \end{cases}$$

Then $u_1 - u_2$ must have its maxima on the boundaries, where $u_1 - u_2 = g(x,t) - g(x,t) = 0$.

Separation of variables for the heat equation

$$\begin{cases} u_t = u_{xx} & x \in [0,1] \\ u(0,t) = u(1,t) = 0 \end{cases}$$

- Search for separated solution

$$u(x,t) = X(x)\,T(t)$$

- Plug into PDE:

$$X(x)\,T'(t) = X''(x)\,T(t)$$

- Note that this yields two decoupled ODE:

$$\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = \text{const} = -\lambda^2$$

- Satisfying initial data is not really possible, but BCs can be solved:

$$u(x,0) = X(x)\,T(0)$$

$$\Rightarrow \frac{u_0''(x)}{u_0(x)} = \text{const} \qquad \text{~~hard but~~ allows only certain ICs}$$

but

$$u(0,t) = X(0)\,T(t) = 0$$
$$u(1,t) = X(1)\,T(t) = 0$$

can be set with $X(0) = X(1) = 0$

- Solutions for X are then

$$X(x) = a\cos(\lambda x) + b\sin(\lambda x)$$

- Impose $X(0) = X(1) = 0$:

$$\Rightarrow a = 0, \qquad b\sin\lambda = 0$$

- Solution is boring for $b = 0$ so impose

$$\sin\lambda = 0$$
$$\lambda = n\pi \qquad n \in \mathbb{Z}$$

- Solution of T: $c e^{-\lambda^2 t} = b_n e^{-(n\pi)^2 t}$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} b_n e^{-(n\pi)^2 t} \sin(n\pi x)$$

let b_n absorb negative integers.

- Initial conditions:

$$u_0(x) = u(x,0) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)$$

– $u_0(x)$ is given by its Fourier series representation

$$\int_0^1 u_0(x) \sin(m\pi x)\,dx = \sum_{n=1}^{\infty} b_n \int_0^1 \sin(m\pi x) \sin(n\pi x)\,dx$$

$$= b_m \int_0^1 \sin^2(m\pi x)\,dx$$

$$= \frac{b_m}{2}$$

$$\Rightarrow \quad b_m = 2\int_0^1 u_0(x) \sin(m\pi x)\,dx$$

Burger's Equation and Cole-Hopf transform

$$u_t + uu_x = \nu u_{xx}$$

Viscous Shocks

$u(x,t) = U(x-ct)$ ansatz $\quad c = $ shock speed $= \frac{u^- + u^+}{2}$

$\rightsquigarrow (U-c)U'(\xi) = U''(\xi) \quad \xi = x - ct$

$\Rightarrow \frac{U^2}{2} - cU = \nu U' + D$

- BCs: $U(-\infty) = u_-$, $U(+\infty) = u_+$

$$\rightsquigarrow \begin{cases} U = u^+ + \dfrac{u^- - u^+}{1 + \exp\left[(u^- - u^+)\frac{\xi}{2\nu}\right]} \\ \xi = \dfrac{2\nu}{u^+ - u^-} \log\left(\dfrac{u^- - U}{U - u^+}\right) \end{cases}$$

- shock strength $u^- - u^+$
- shock width $\frac{\nu}{u^- - u^+}$

- Diffusivity spreads out the shock.

Cole-Hopf transformation

- Nonlinear transform viscous Burger's into heat equation.

$$\begin{cases} u_t + uu_x = \nu u_{xx} \\ u(x,0) = u_0(x) \end{cases}$$

- Set $u = \phi_x$ and integrate to remove nonlinear term:

$\rightsquigarrow \phi_{xt} + \phi_x \phi_{xx} = \nu \phi_{xxx}$

$\phi_t + \frac{\phi_x^2}{2} = \nu \phi_{xx}$ (Integration by parts).

- Remove nonlinear term with $\phi = -2\nu \log \psi$

$\rightarrow \phi_t = -\frac{2\nu}{\psi}\psi_t \quad \phi_x = -\frac{2\nu}{\psi}\psi_x \quad \phi_{xx} = -\frac{2\nu}{\psi}\psi_{xx} + 2\nu\left(\frac{\psi_x}{\psi}\right)^2$

yields $\psi_t = \nu \psi_{xx}$