Learning Propagators for Sea Surface Height Forecasts Using Koopman Autoencoders

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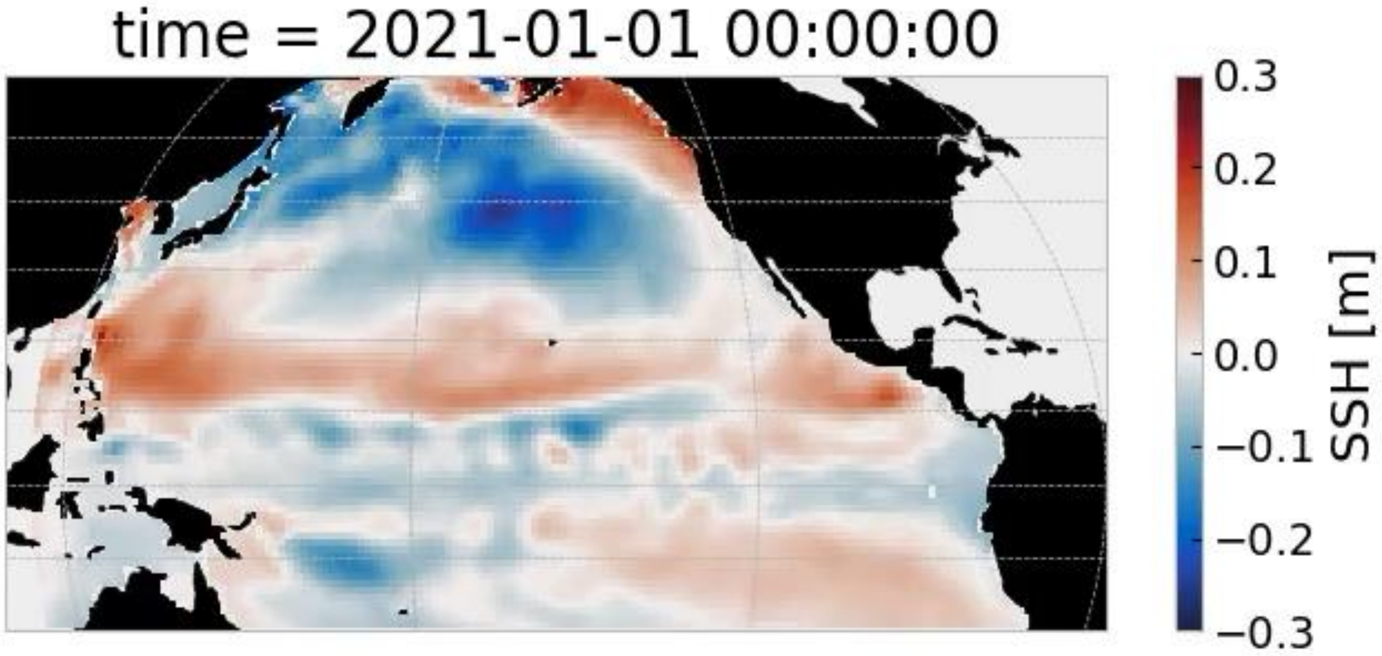
https://m2lines.github.io/

Numerous sources of uncertainty for sea surface height forecasts

Simulated SSH anomalies in CESM2-LE



Sea level components

$$\eta = \frac{p_b - p_a}{\rho_0 g} - \frac{1}{\rho_0}\int_{-H}^{0} \rho \, dz$$

manometric

- surface wind stress
- inverse barometer
- Kelvin waves
- Rossby waves
- advection

steric

- surface heat flux
- heat transport

Statistical-dynamical approaches: Linear inverse modeling (LIM)

- Assume system state is governed by a linear, stochastic dynamical system:

$$\frac{dz}{dt} = \mathbf{A}z + \xi; \quad \xi_i \sim N(0, \sigma_i^2)$$

- Best estimate for $\mathbf{A}$:

$$\mathbf{A} = \frac{1}{\tau_0} \log[\mathbf{C}(\tau_0)\mathbf{C}(0)^{-1}]$$

where the covariance matrix $\mathbf{C}$ is given by $\mathbf{C}_{ij}(t+\tau) = \langle z_i(t+\tau) z_j(t) \rangle$.

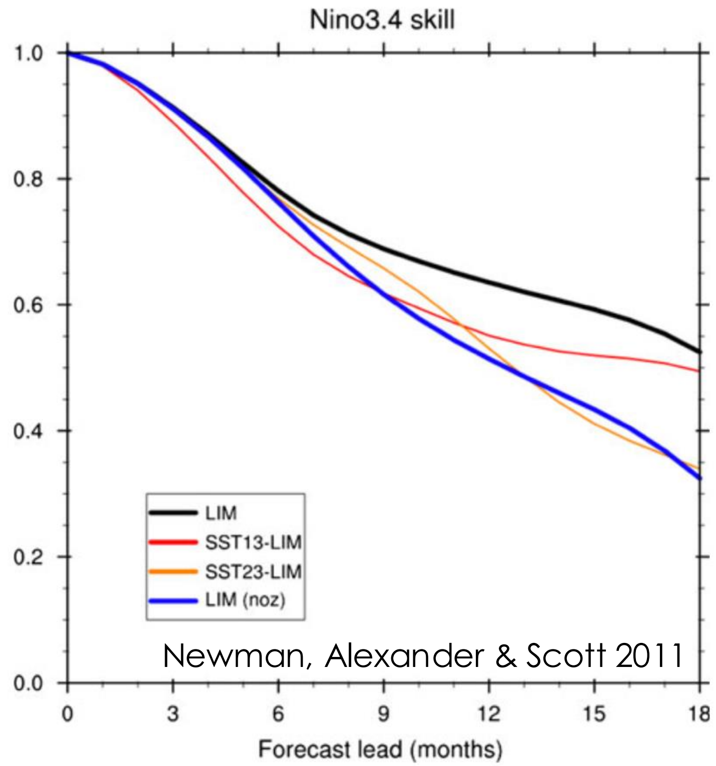
- Forecasts at lag τ are given by

$$\hat{z}_{n+\tau} = e^{\mathbf{A}\tau} z_n$$

Challenges with LIM

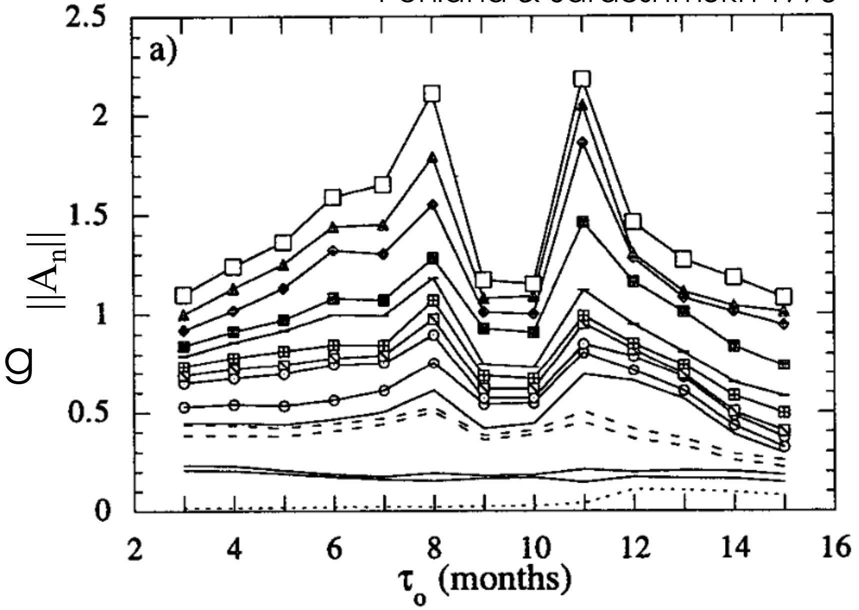
- **Dimensionality reduction**: Computing $\mathbf{C}(\tau_0)\mathbf{C}(0)^{-1}$ requires low-dimensional state variable z

- **Nonlinear dynamics**: highly nonlinear dynamics may not be well-represented using a linear propagator



Worse long-term predictions with more PCs than with fewer PCs!!

Penland & Sardeshmukh 1995

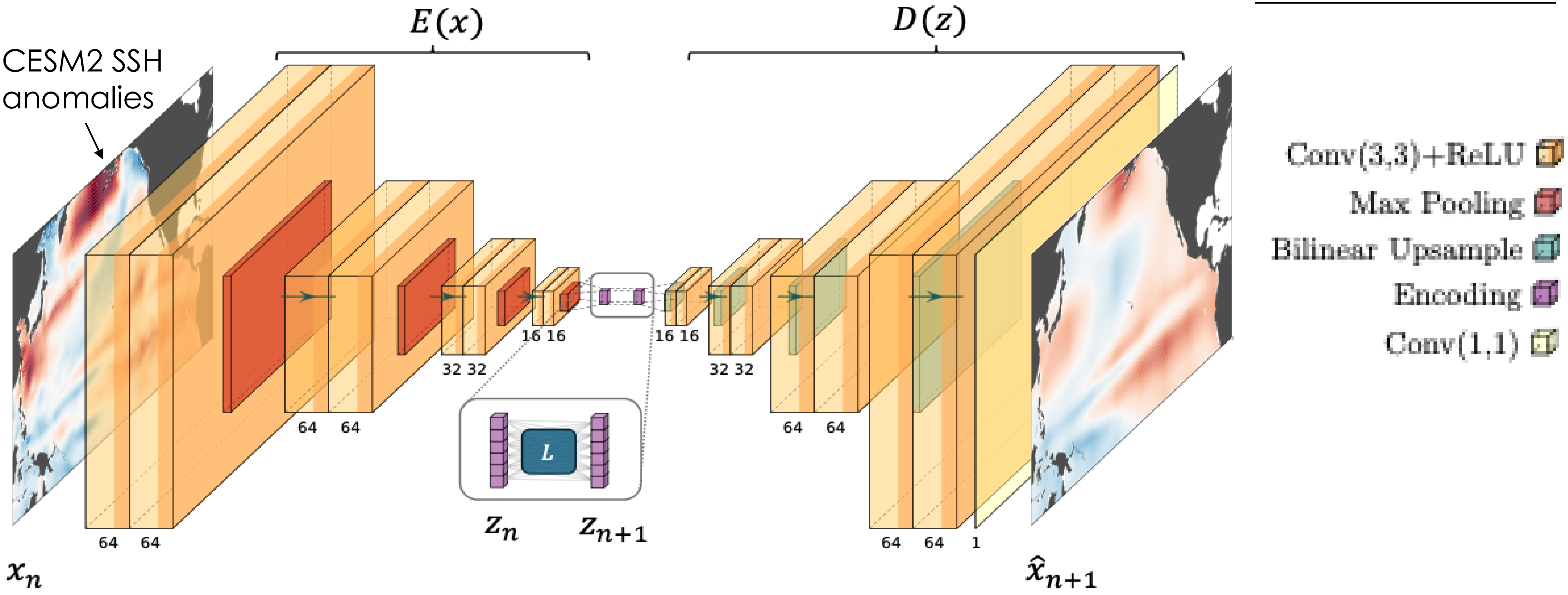


Linear propagator depends on time lag

Goals

- Can we perform dimensionality reduction in a way that results in better forecasts of SSH on daily-to-interannual timescales?
- Can we ensure that nonlinear dynamics are well-modelled by a linear propagator?

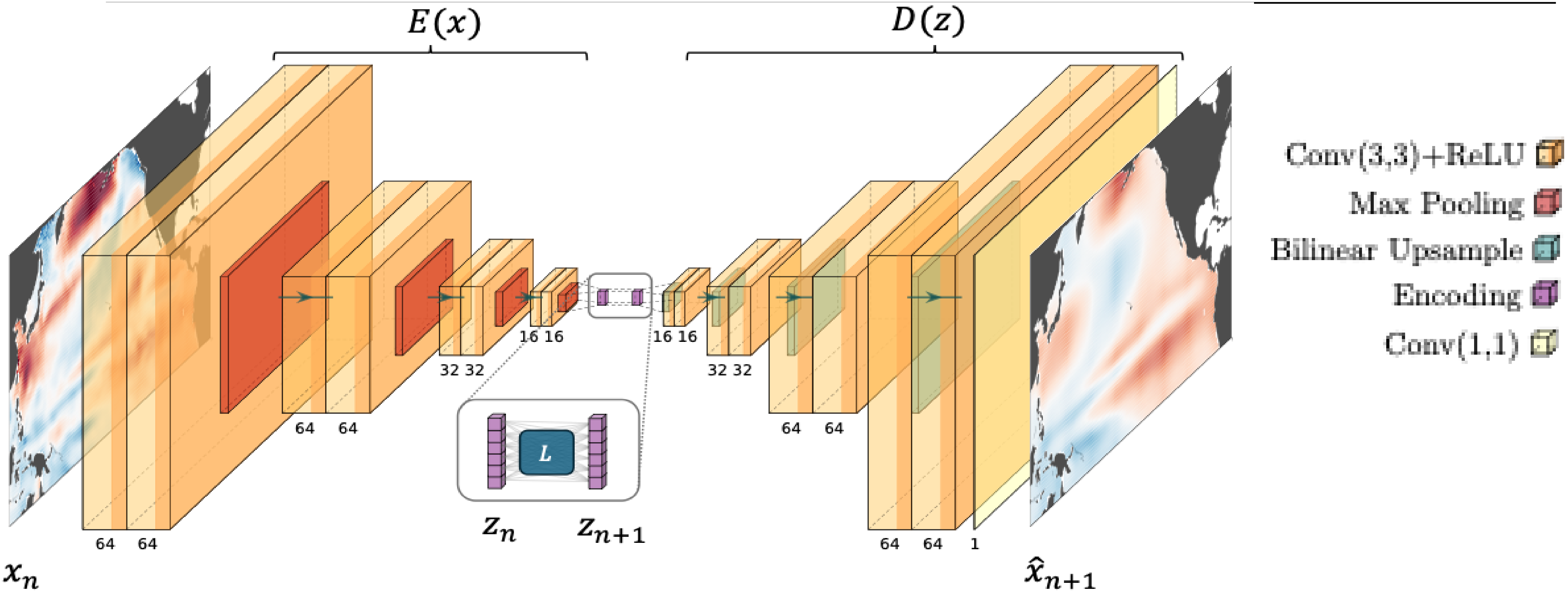
Approach: Koopman Autoencoder



Timestepping and dimensionality reduction are learned together!

e.g. Lusch, Kutz & Brunton 2018

Approach: Koopman Autoencoder



Loss functions

1. **Reconstructions** $\mathcal{L}_{\text{reconst}}(x_n) = \|x_n - D(E(x_n))\|_{2,w}^2$
2. **Predictions** $\mathcal{L}_{\text{pred}}(x_n, \ldots, x_{n+k}) = \frac{1}{k}\sum_{\ell=1}^{k} \|x_{n+\ell} - D(L^\ell E(x_n))\|_{2,w}^2$
3. **Linearity** $\mathcal{L}_{\text{linear}}(x_n, x_{n+1}) = \|LE(x_n) - E(x_{n+1})\|_2^2$

Baselines

Compare Koopman Autoencoder to baselines when dimensionality reduction and propagators are learned separately:

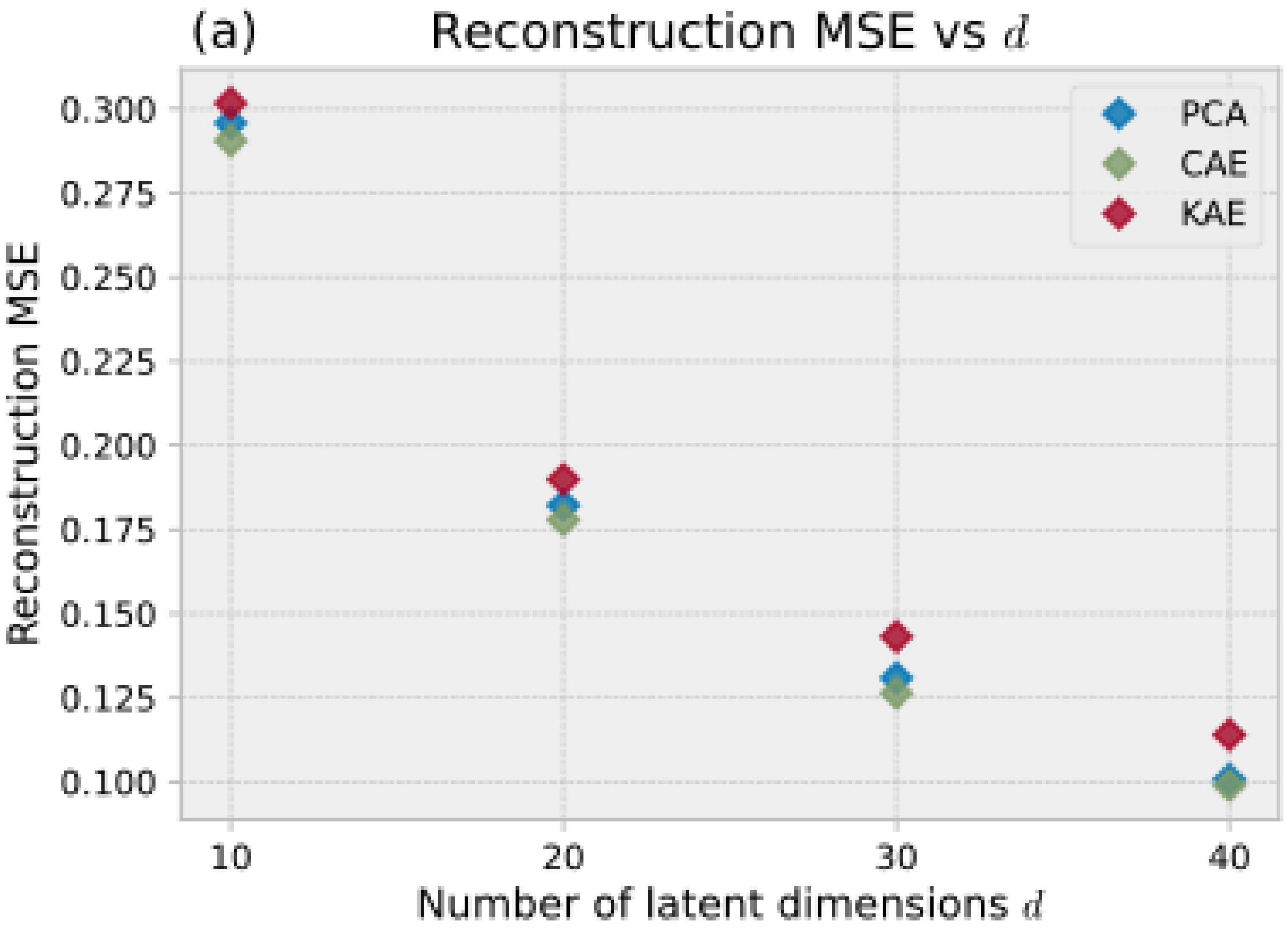
Dimensionality reduction techniques

1. Principal Component Analysis (PCA)
2. Convolutional Autoencoder (CAE)
   - Same architecture as the Koopman Autoencoder, but without linear embedding

Latent-space propagators

1. Damped persistence (DP)
2. Linear inverse modeling (LIM)

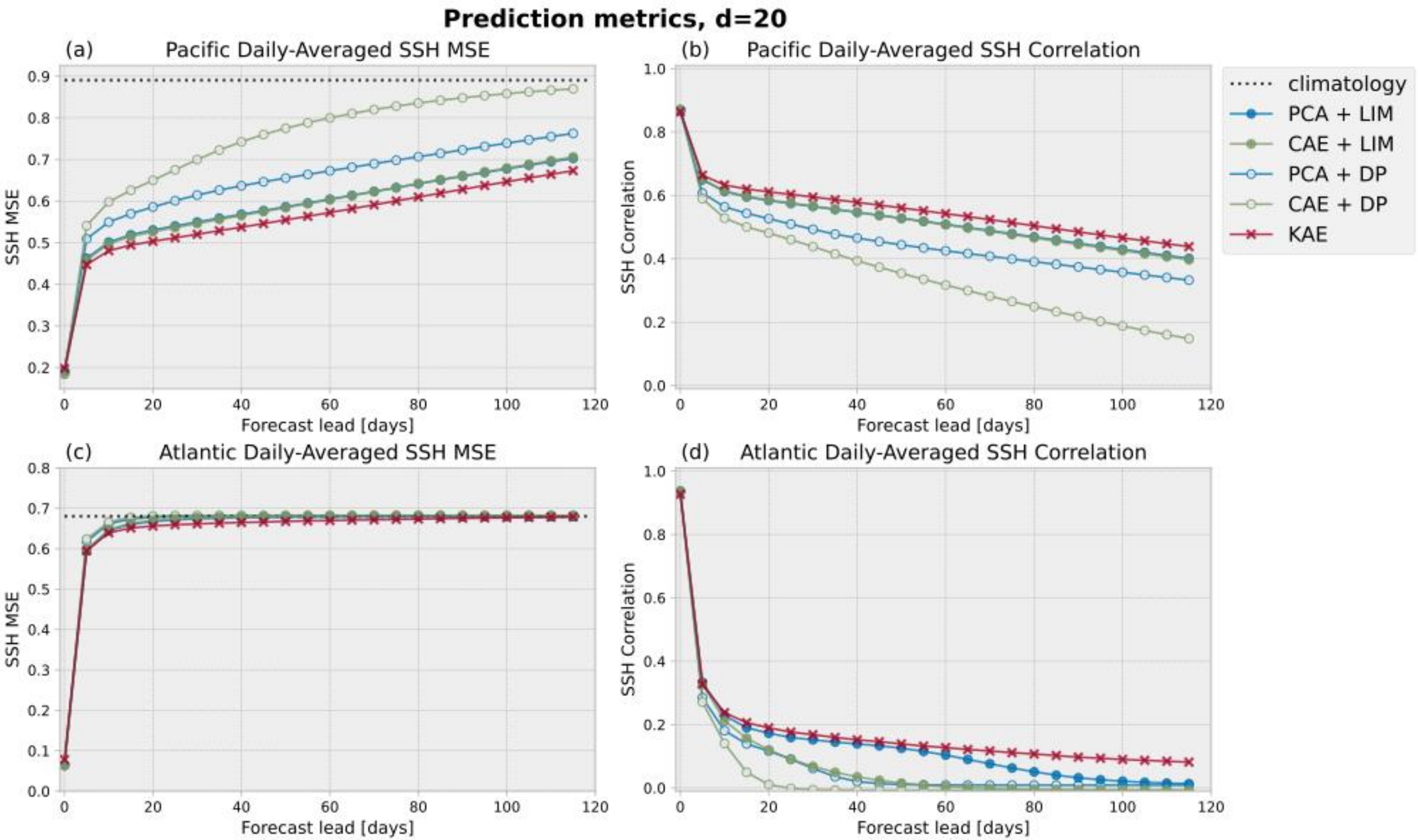
Sensitivity to dimensionality of propagator



- Reconstruction performance improves with dimensionality
- CAE has lowest reconstruction MSE
- Koopman Autoencoder has the lowest reconstruction error

Performance metrics

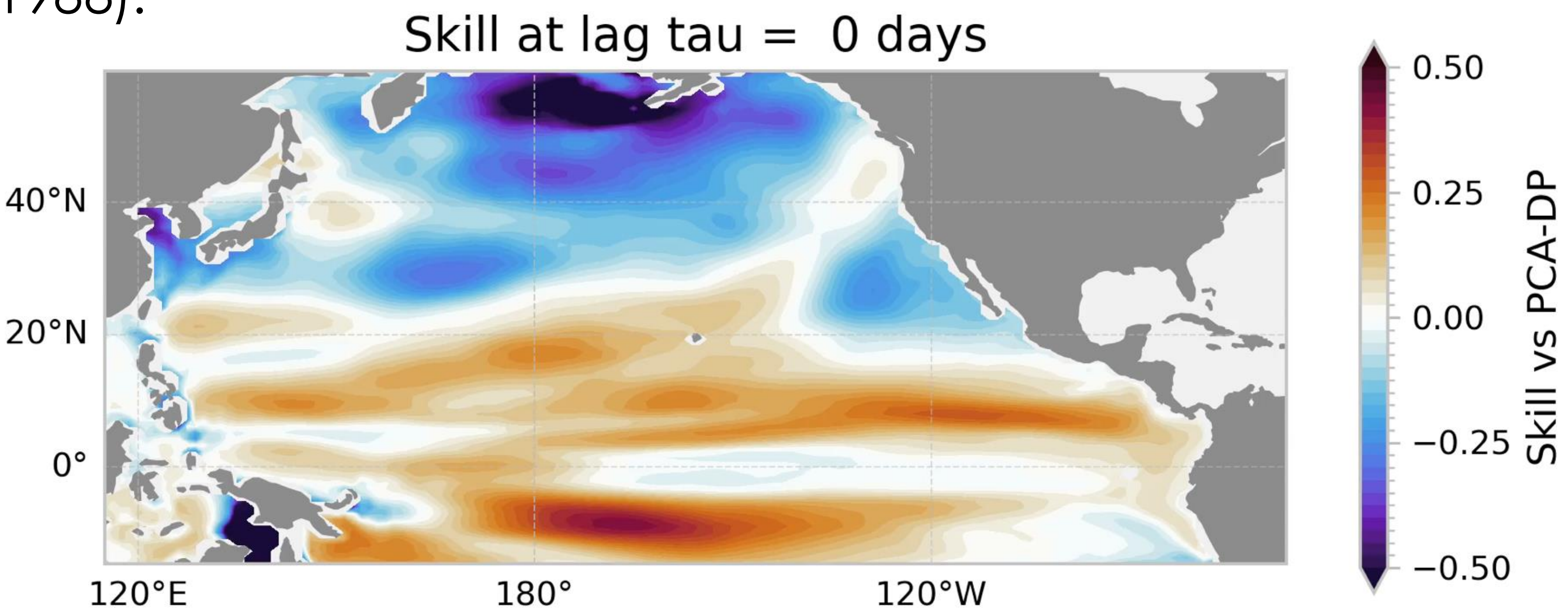
Colors: dimensionality reduction technique
Markers: propagator type



Regions of skill

Skill of Koopman Autoencoder relative to PCA+DP (Murphy, 1988):

$$SS = 1 - \frac{\mathrm{MSE}_{\mathrm{KAE}}}{\mathrm{MSE}_{\mathrm{PCA+DP}}}$$

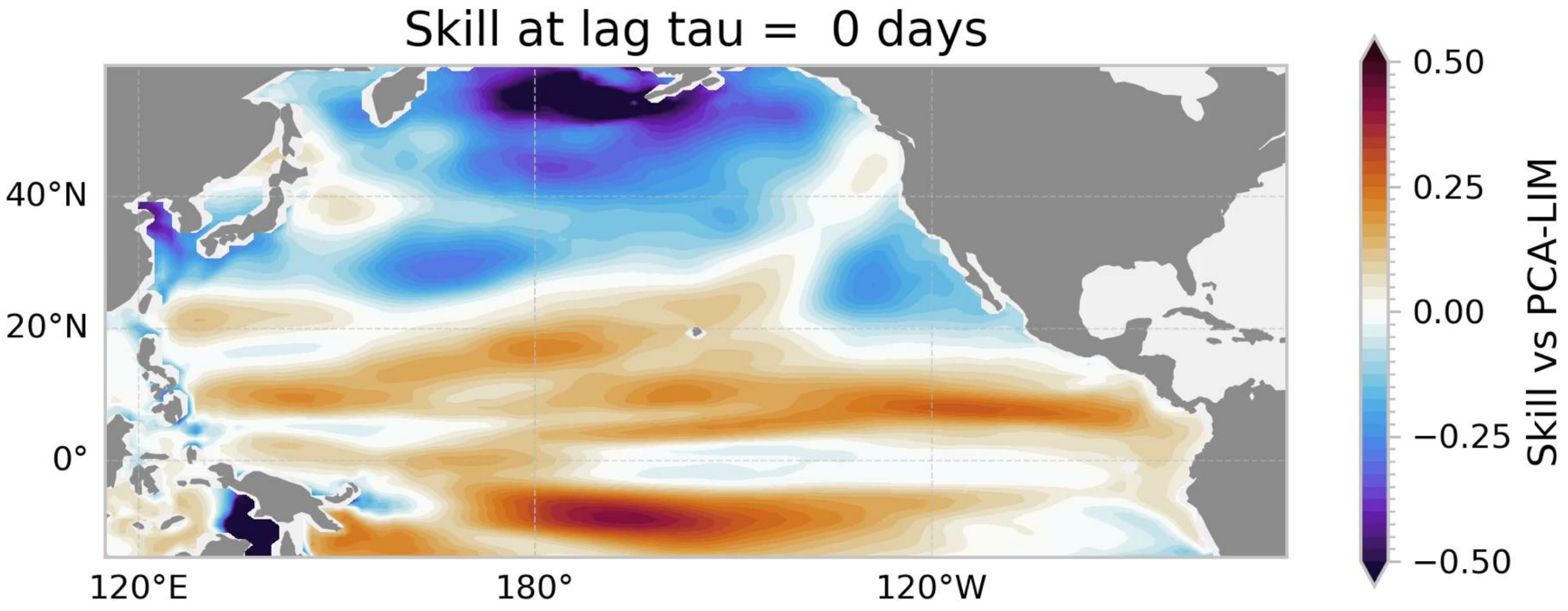


SS ≈ 1: high skill relative to baseline
SS ≈ 0: low skill

Regions of skill

Skill of Koopman Autoencoder relative to PCA+LIM:

$$SS = 1 - \frac{\mathrm{MSE}_{\mathrm{KAE}}}{\mathrm{MSE}_{\mathrm{LIM}}}$$



Conclusions

- Learning dimensionality reduction and a propagator simultaneously in the Koopman Autoencoder results in a better propagator for SSH forecasts
- Reconstruction skill of the model identifies regions in which improved representation can result in better regional predictions

References

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2. Penland, C., & Sardeshmukh, P. D. (1995). The optimal growth of tropical sea surface temperature anomalies. *Journal of Climate*, 8(8), 1999–2024.
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4. Koopman, B. O. (1931). Hamiltonian systems and transformation in Hilbert space. *Proceedings of the National Academy of Sciences*, 17(5), 315–318.