Quantifying changes in probability distributions of sea level from tide gauge data

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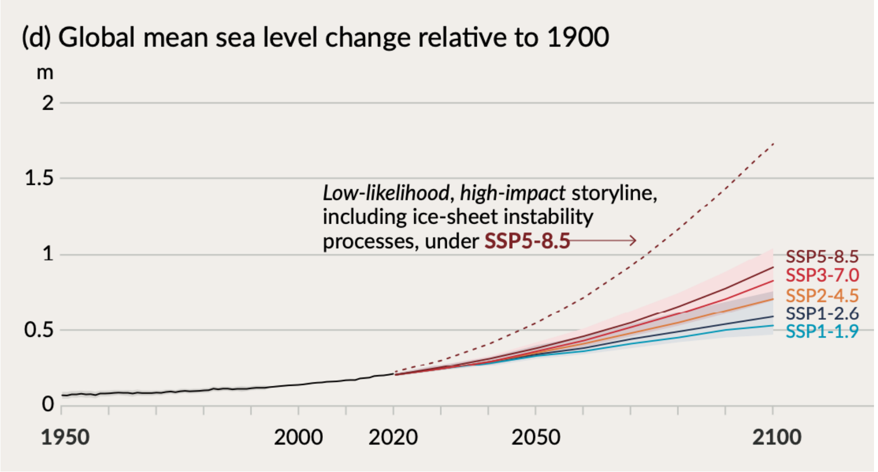
Student Probability Seminar
8 March 2022

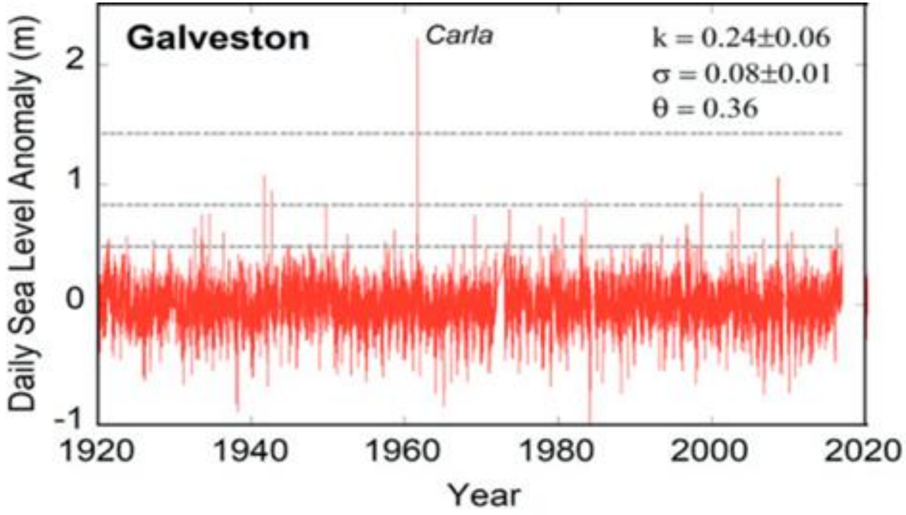
Sea level change and variability

Long-timescale changes



IPCC (2022)

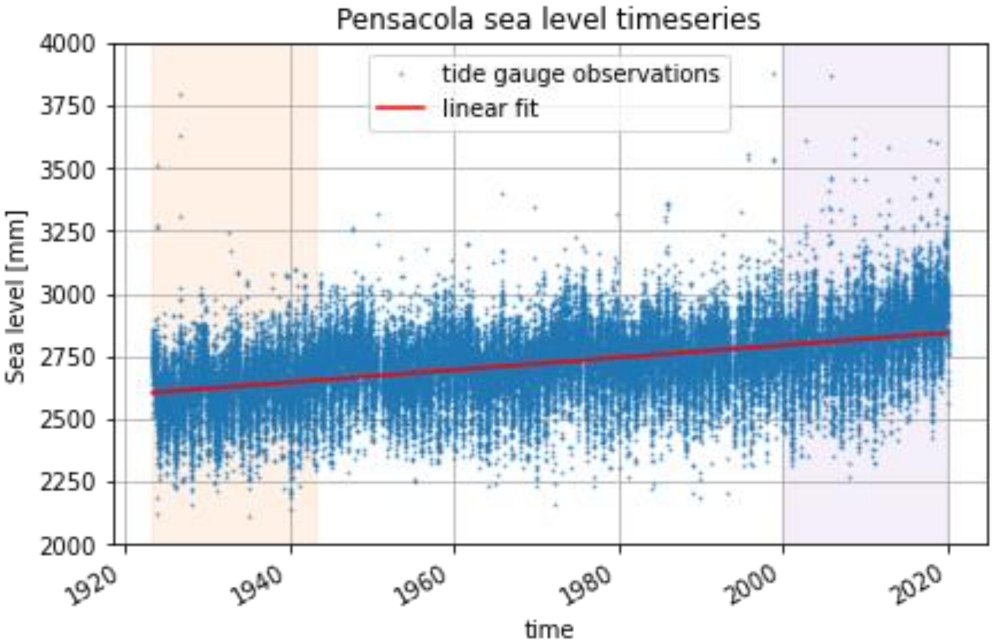
Short-timescale variability



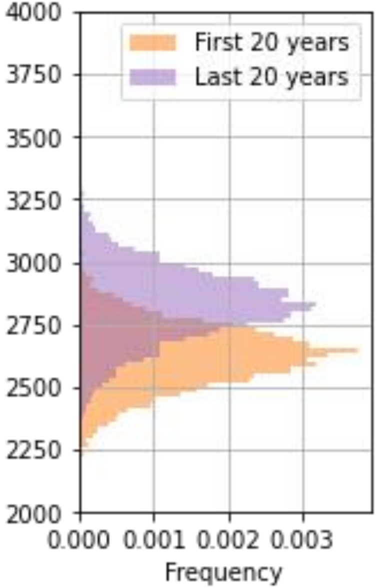
Yin et al (2020)

Predicting future sea level events

Requires understanding two components:



1. Changes in **local mean**



2. Changes in the **shape of distributions**

Research goals:

1. Quantify changes in probability distributions of observed sea level
2. Interpret changes in distribution in terms of the statistical moments

Methods: quantifying changes in sea level distributions

Extending approach of **McKinnon and Rhines 2016**:

- Quantile regression
- Projection onto basis functions

Observed changes in distributions

Quantile regression:

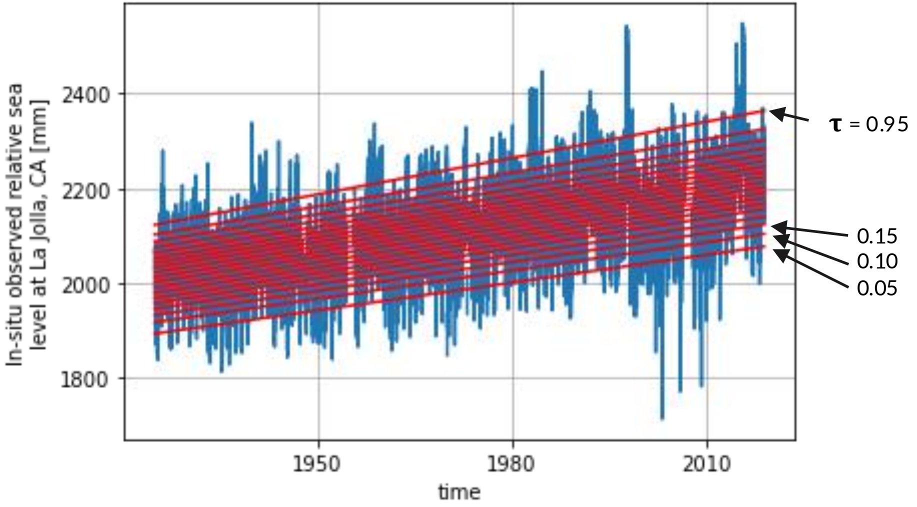
- Ordinary least squares regression seeks to estimate the conditional **mean** of a response variable Y to a predictor X:

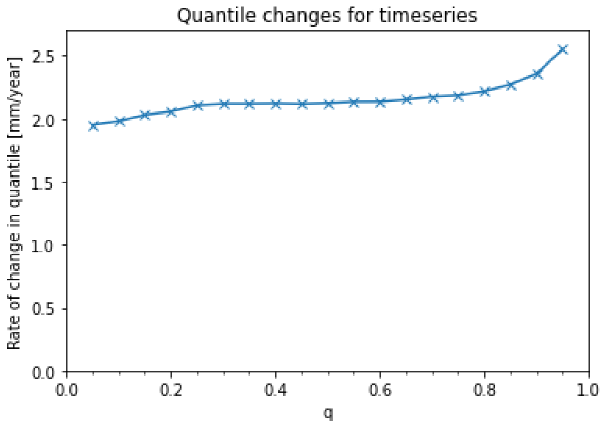
$$\mathbb{E}[Y|X] = \beta_0 + \beta_1 X$$

- Quantile regression seeks to estimate conditional **quantiles** $Z(q)$ of a response variable Y to a predictor X:

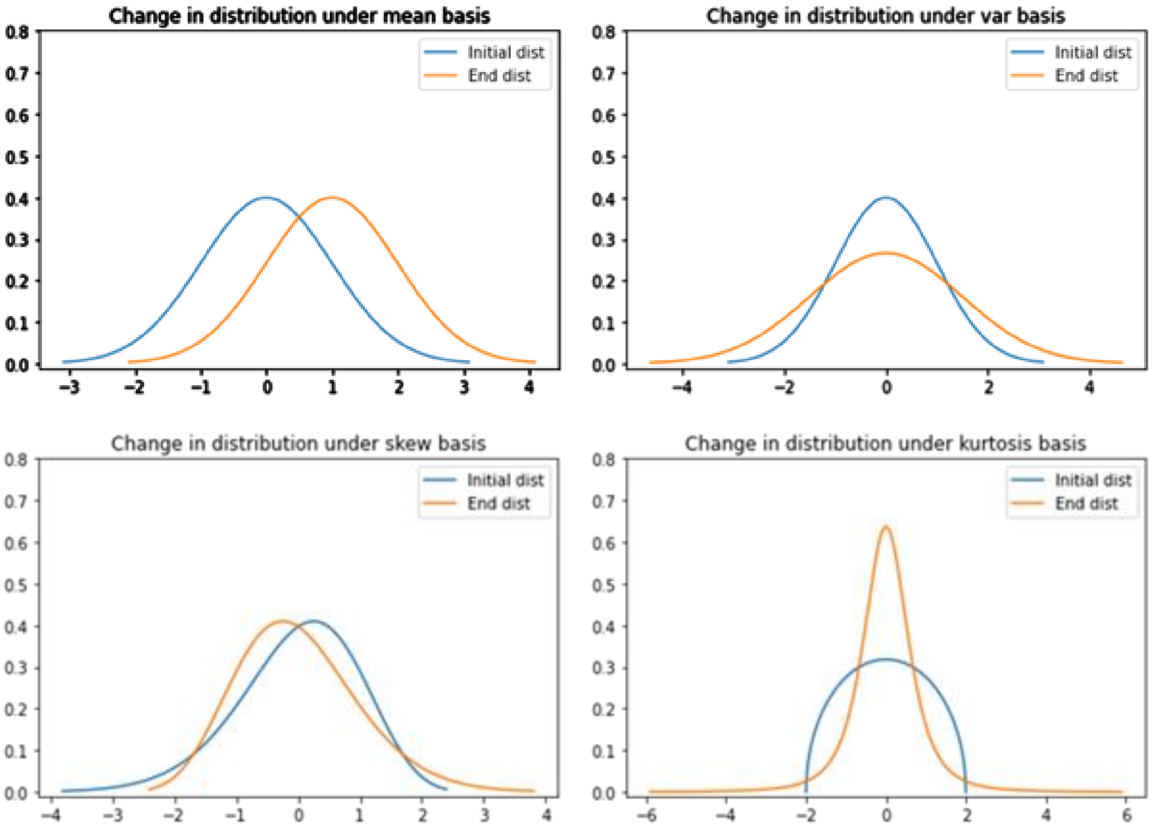
$$Z_{Y|X}(q) = \beta_0(q) + \beta_1(q) X$$

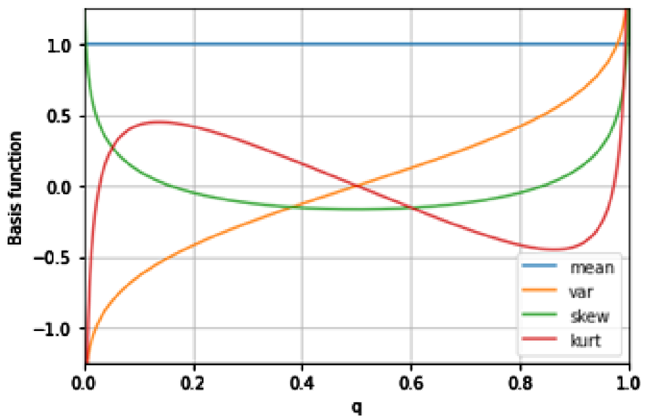
Quantile regression on a daily tide gauge timeseries



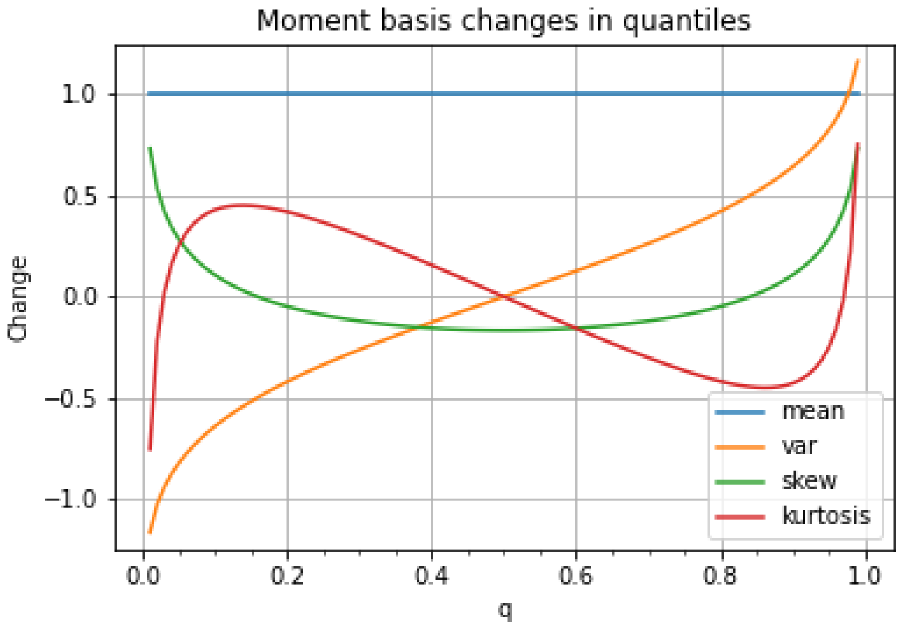


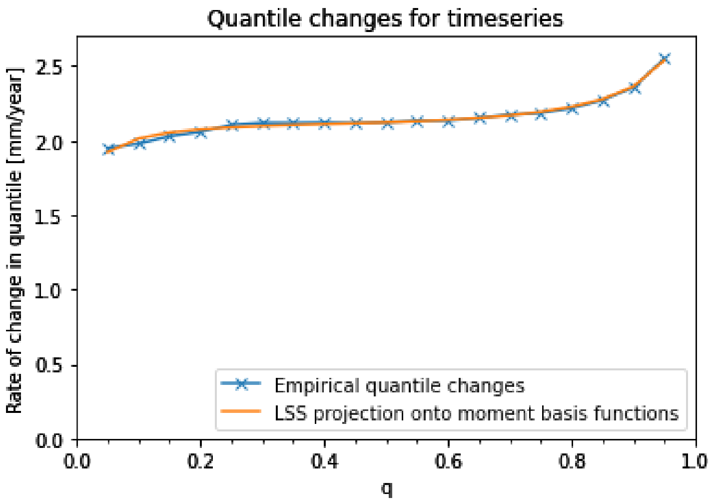
Projection onto basis functions





Projection onto basis functions





$$v \approx c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4$$

$$[c_i] = \mathrm{mm/year}$$

$$w_i = \frac{c_i^2}{\|\boldsymbol{c}\|^2}$$

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weights: [0.93  0.042 0.012 0.016]
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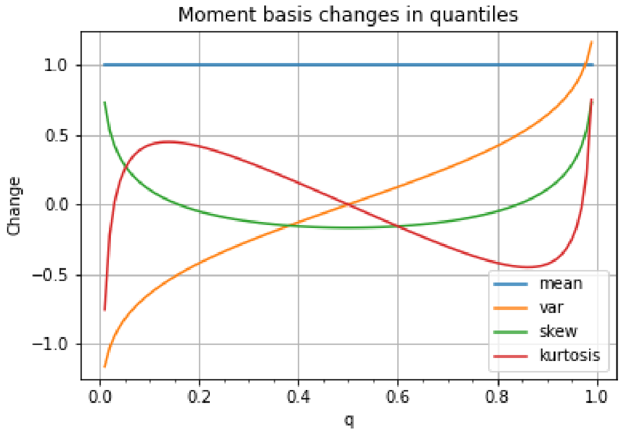
Motivation for Legendre basis

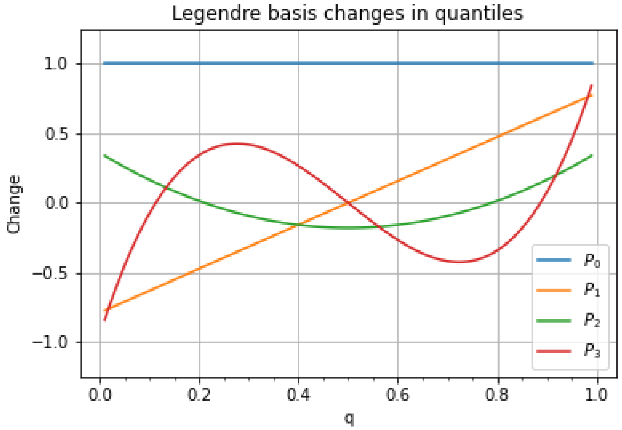
Moment basis functions are correlated:

	mean	variance	skew	kurtosis
mean	1.00	-0.23	0.15	0.35
variance	-0.23	1.00	0.00	-0.90
skew	0.15	0.00	1.00	-0.00
kurtosis	0.35	-0.90	-0.00	1.00

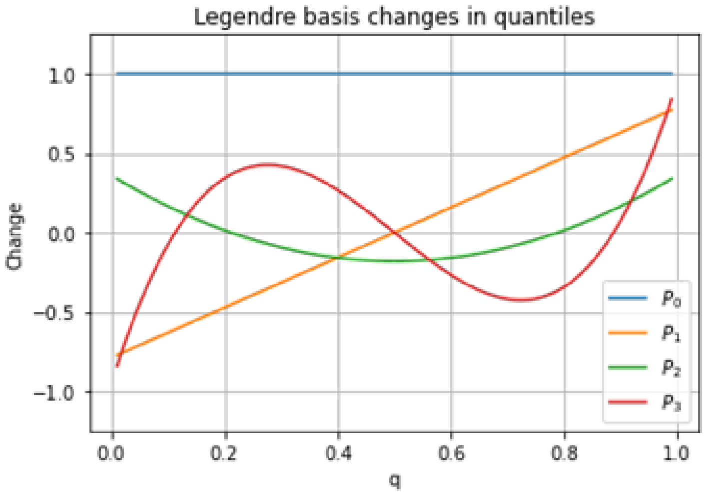


Introduce Legendre basis:

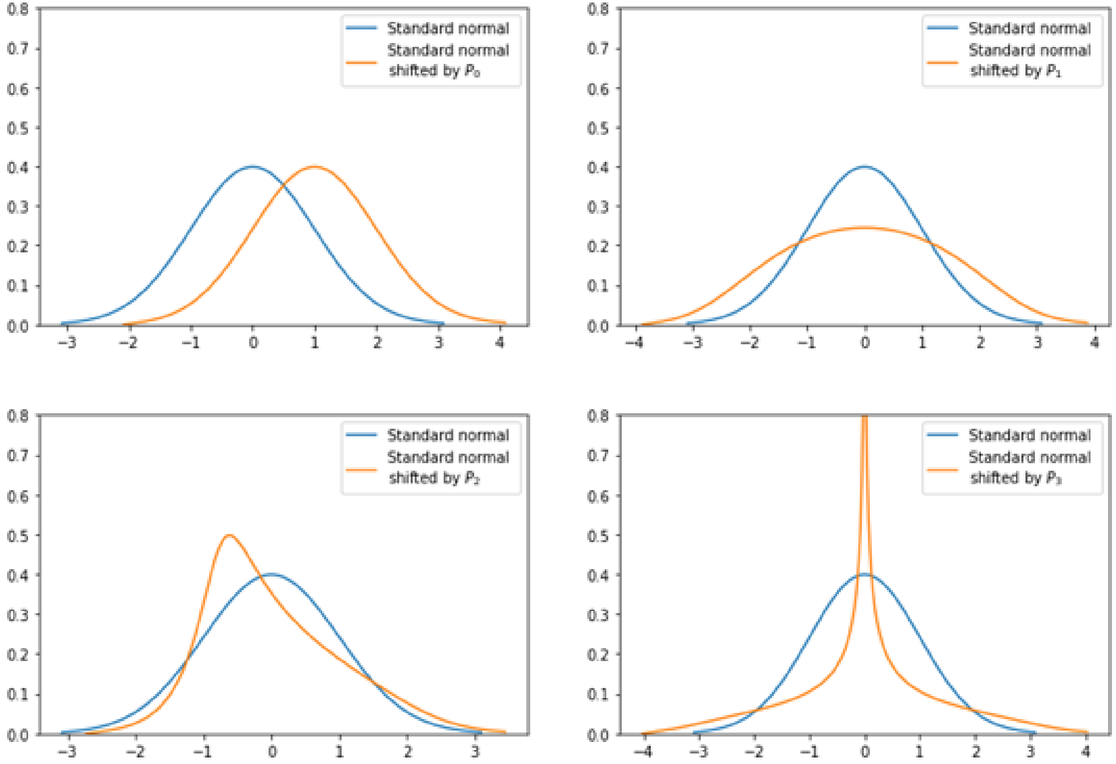
	mean	variance	skew	kurtosis
mean	1.00	0.00	-0.00	0.00
variance	0.00	1.00	0.00	-0.00
skew	-0.00	0.00	1.00	0.00
kurtosis	0.00	-0.00	0.00	1.00



Legendre basis effects on distributions



Effect of Legendre basis applied to standard normal distribution



Cornish-Fisher Expansion

- **Is there a relationship between the moments of a distribution and its quantiles? Yes:**
- Cornish and Fisher (1937) show that the operator $\exp\left\{\frac{a_r}{r!}\left(-\frac{d}{dx}\right)^r\right\}$ increases the r^{th} cumulant of a distribution f(x), but leaves the distribution otherwise unchanged
- This allows us to estimate percentiles y_p of a distribution given the cumulants:

$$y_p \approx \mu + \sigma w;$$

$$w = z_q + \left(z_q^2 - 1\right)\frac{S}{6} + \left(z_q^3 - 3z_q\right)\frac{K}{24} - \left(2z_q^3 - 5z_q\right)\frac{S^2}{36}$$

where z_q is the q-quantile of the standard normal distribution.

- Here S and K are parameters which approximate the skewness and kurtosis for small deviations from Gaussianity.

Changes in percentiles from changes in moments

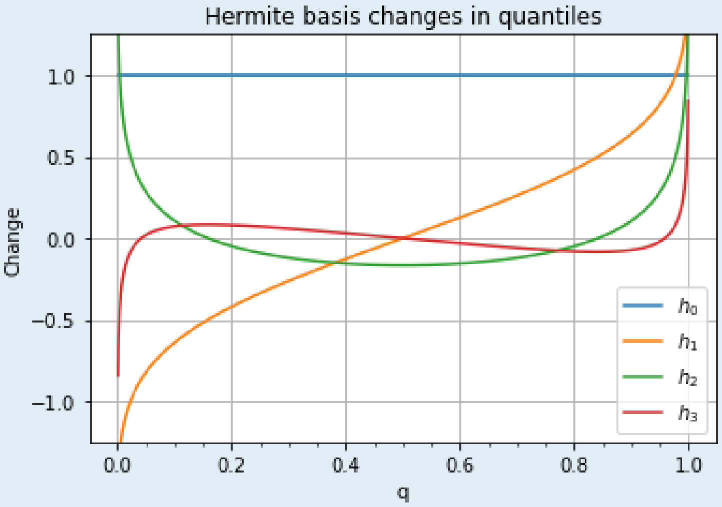
$$y_p \approx \mu + \sigma w;$$
$$w = z_q + \left(z_q^2 - 1\right)\frac{\gamma}{6} + \left(z_q^3 - 3z_q\right)\frac{\kappa}{24} - \left(2z_q^3 - 5z_q\right)\frac{\gamma^2}{36}$$

Take derivatives and evaluate near standard normal $\boldsymbol{k} = (\mu, \sigma^2, \gamma, \kappa) = (0, 1, 0, 0)$:

$$\left.\frac{\partial y_p}{\partial \mu}\right|_{\boldsymbol{k}} = 1$$
$$\left.\frac{\partial y_p}{\partial \sigma^2}\right|_{\boldsymbol{k}} = \frac{1}{2} z_q$$
$$\left.\frac{\partial y_p}{\partial \mu}\right|_{\boldsymbol{k}} = \frac{1}{6}(z_q^2 - 1)$$
$$\left.\frac{\partial y_p}{\partial \mu}\right|_{\boldsymbol{k}} = \frac{1}{24}(z_q^3 - 3z_q)$$

These are Hermite polynomials of z_q – **and form an orthogonal set of functions** on $L^2[0,1]$

Hermite basis effects on distributions



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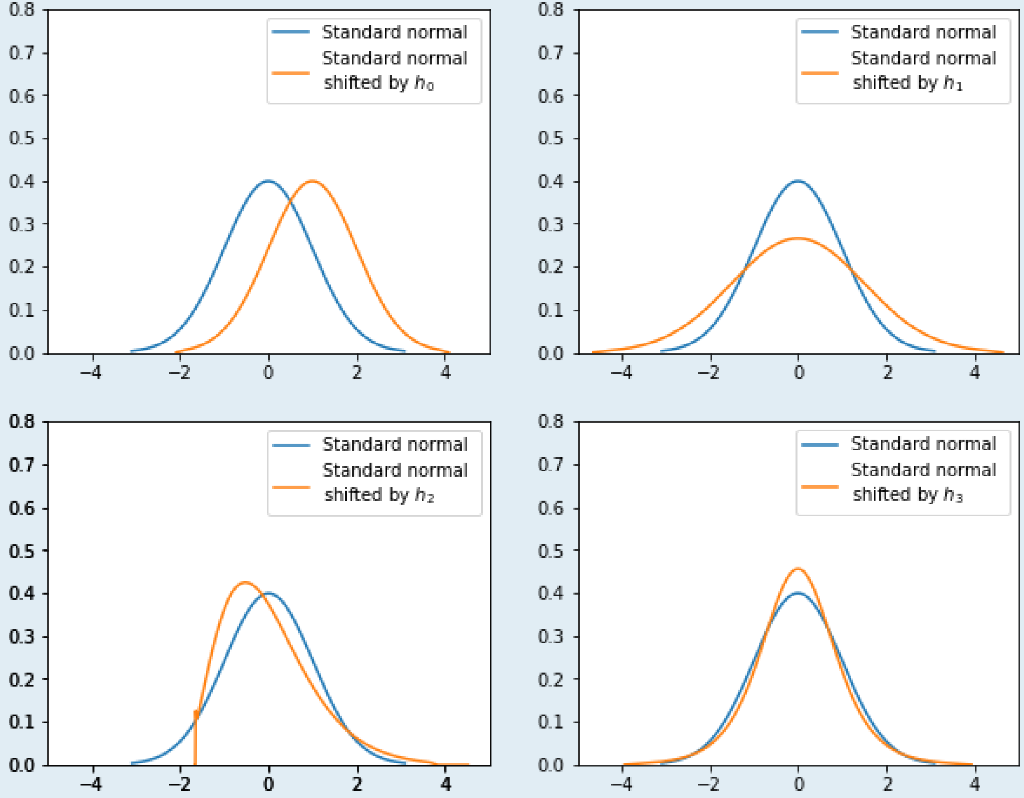
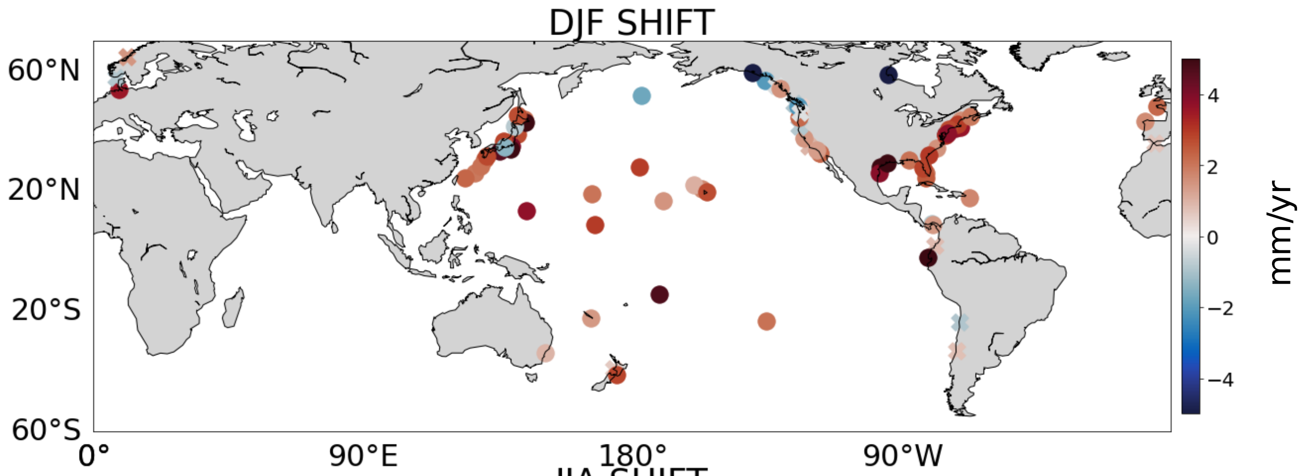


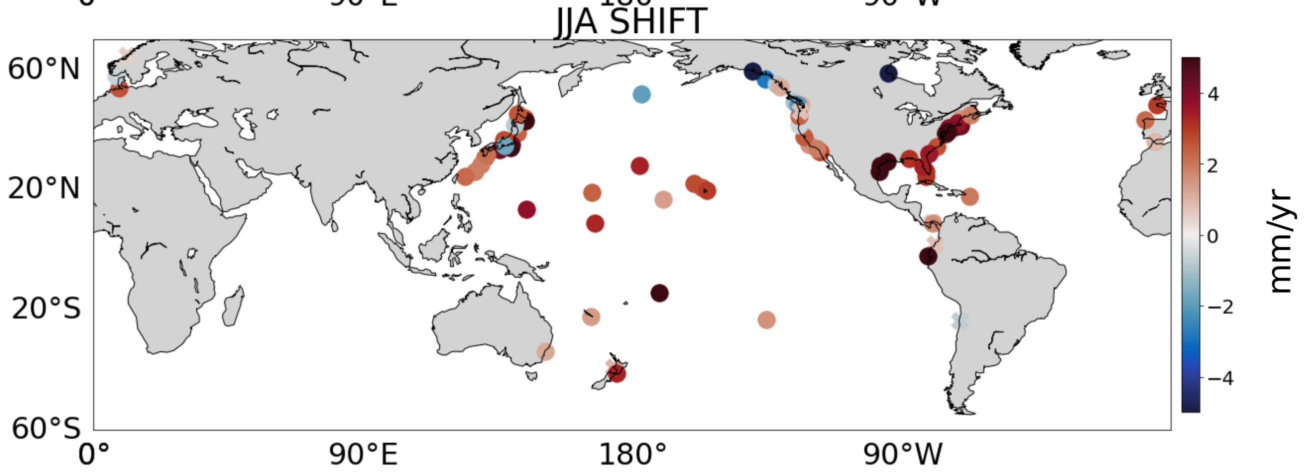
Results: application to tide gauges

- Daily sea level from 80 tide gauges from 1970-2018
- Divide into winter and summer seasons (DJF and JJA)
- Statistical significance via block bootstrapping

Changes in mean sea level

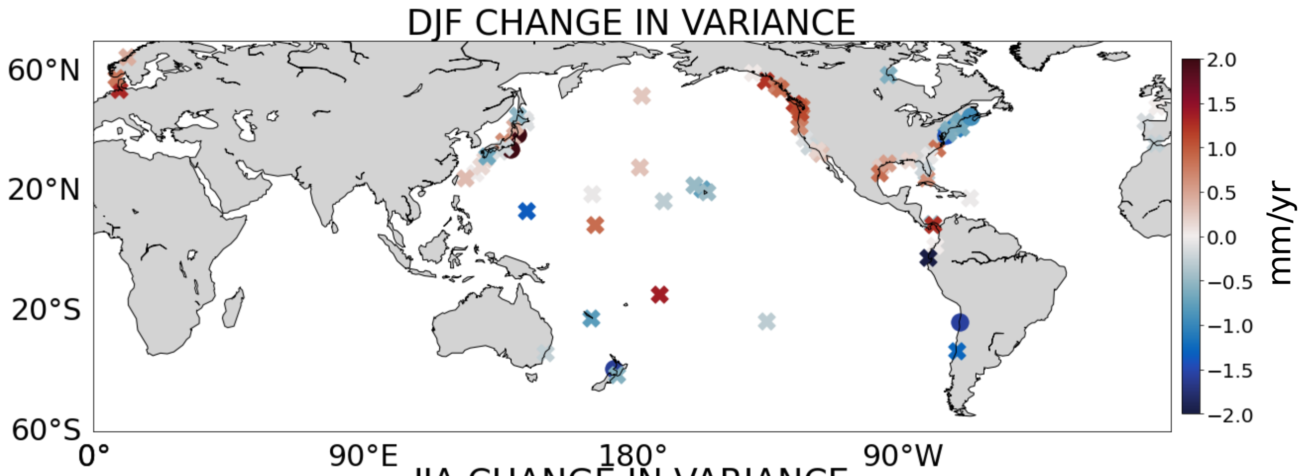


Statistical significance for 67/80 tide gauges

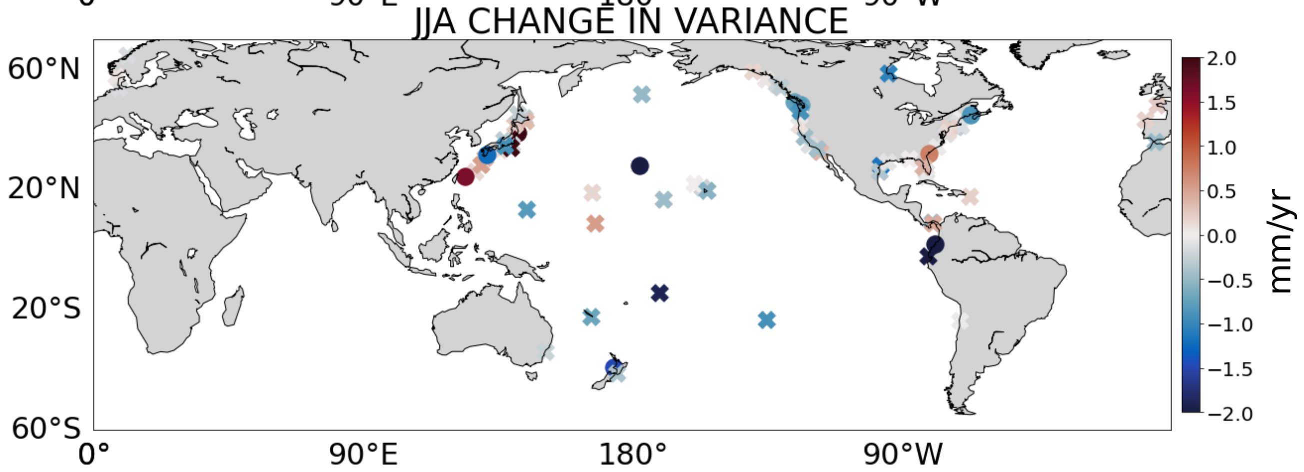


Statistical significance for 73/79 tide gauges

Changes in sea level variance



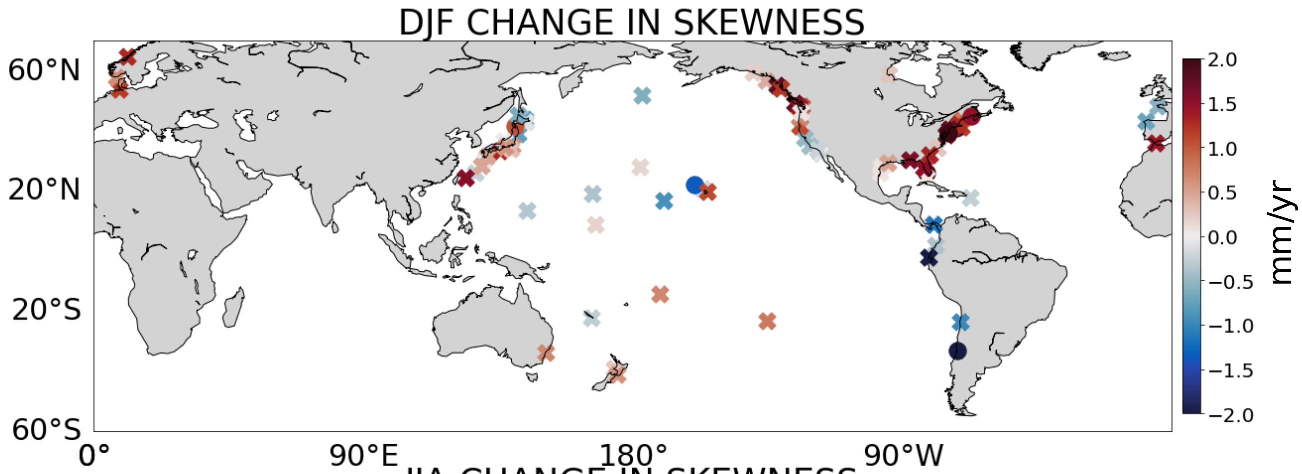
Statistical significance for 6/80 tide gauges



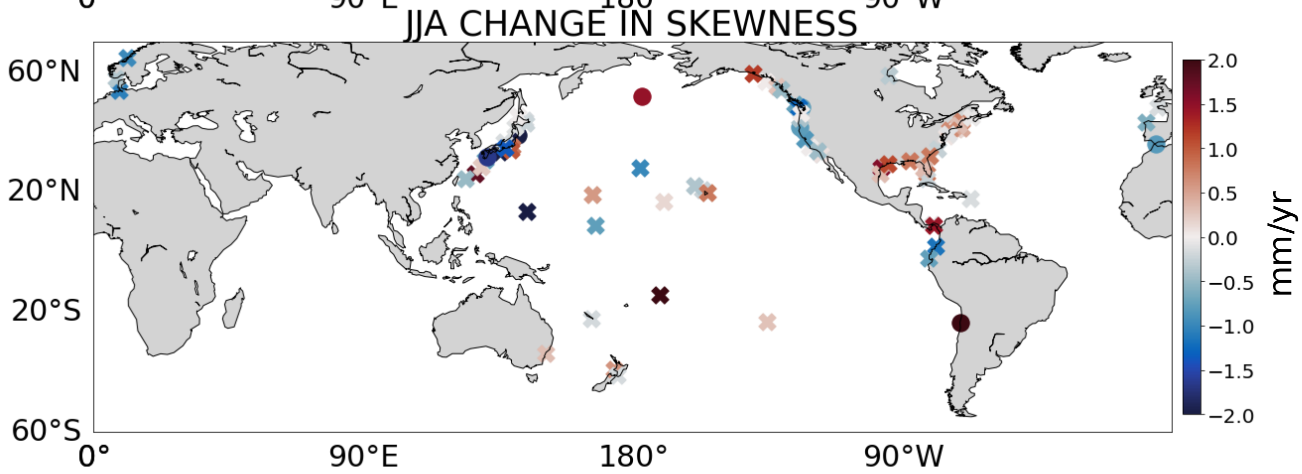
Statistical significance for 10/79 tide gauges

Changes in sea level skewness

= statistically significant
= lacks significance



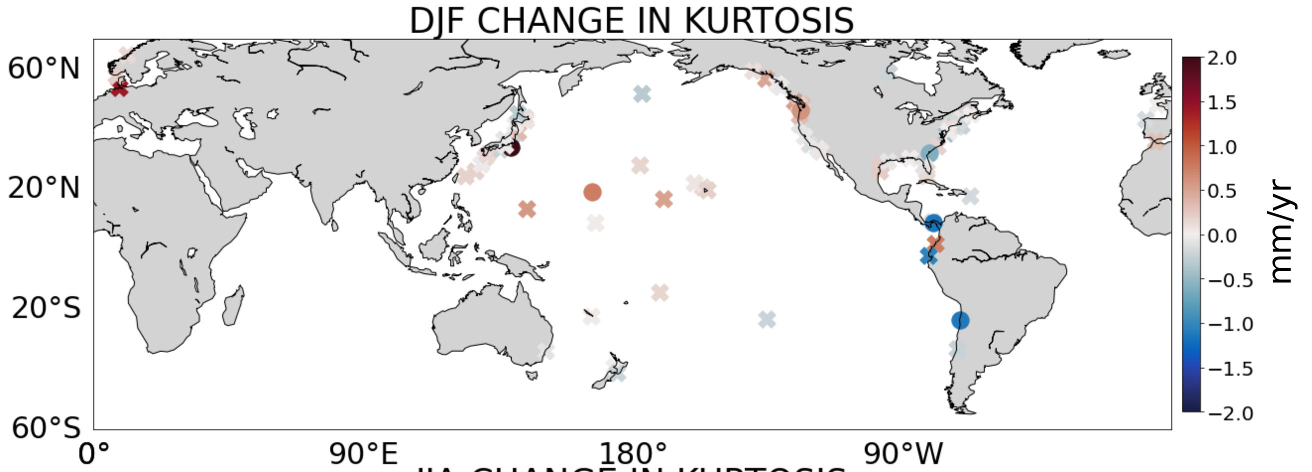
Statistical significance for 9/80 tide gauges



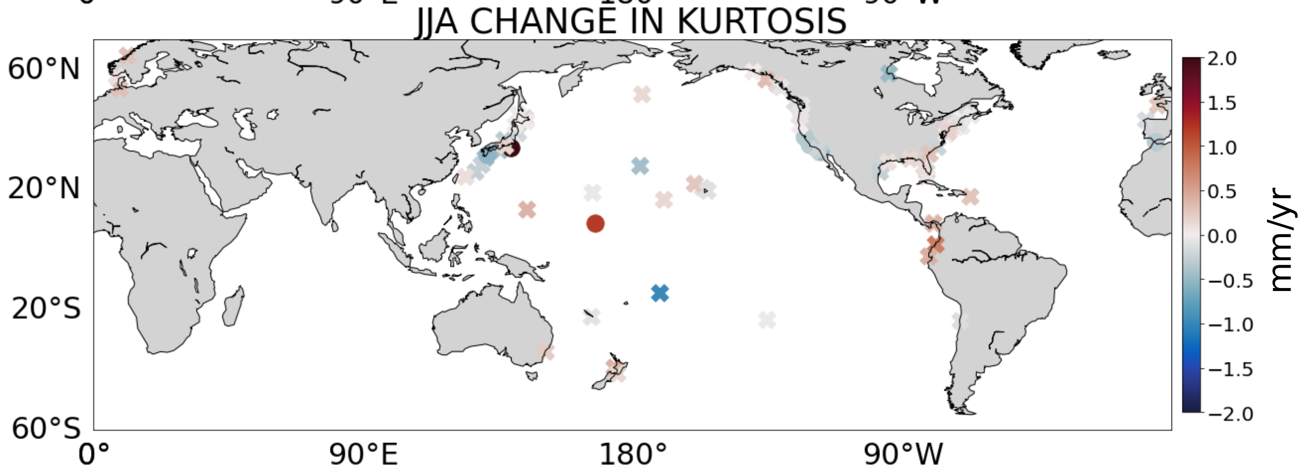
Statistical significance for 7/79 tide gauges

Changes in sea level kurtosis

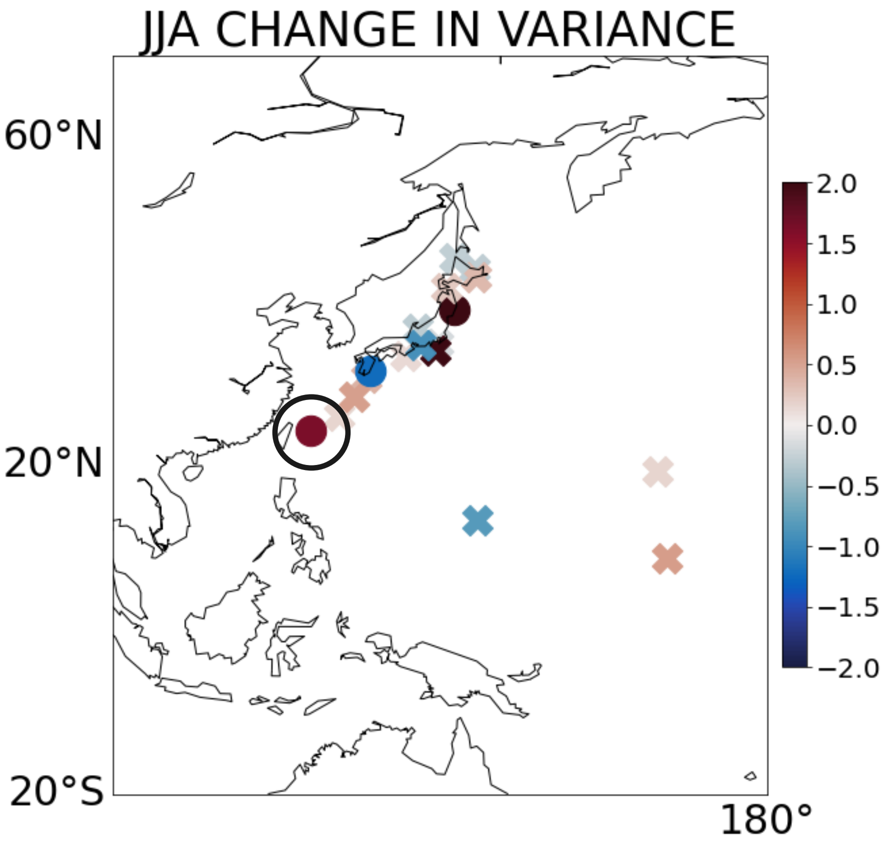
● = statistically significant
✖ = lacks significance



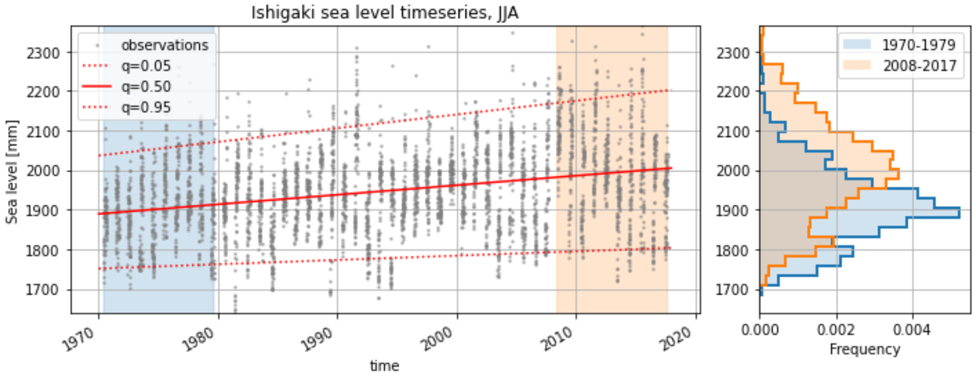
Statistical significance for 6/80 tide gauges

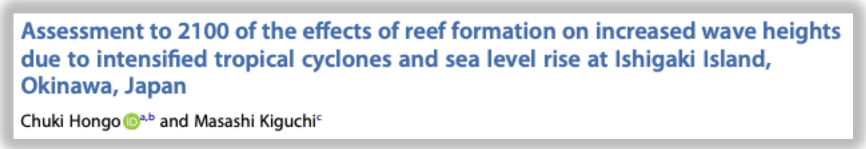


Statistical significance for 7/79 tide gauges



Example tide gauge with statistically significant changes in mean and variance





Conclusions

1. Dominant change in sea level distributions is due to changes in the mean, consistent with previous studies
2. Some tide gauges show statistically significant changes in higher order moments as well, indicating that changing probability distributions can be important

References

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