Ocean Dynamics Cheat Sheet

Primitive Equations

$$\begin{cases} \frac{Du}{Dt} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} \\ \frac{Dv}{Dt} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y} \\ \frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g \end{cases}$$

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{v}) = 0$$

$$\begin{cases} \frac{DT}{Dt} = F_T - Q_T \\ \frac{DS}{Dt} = F_S - Q_S \end{cases} \quad \text{or} \quad \frac{D\rho_\theta}{Dt} = \frac{D}{Dt}\left(\delta\rho - \frac{\delta p}{c_s^2}\right) = 0$$

Equation of state: $\rho = \rho_0\left[1 - \beta_T \Delta T + \beta_S \Delta S + \beta_p \Delta p\right]$

Boussinesq Equations

Continuity: $\nabla\cdot\vec{v} = 0$

Momentum: $\frac{D\vec{u}}{Dt} + \vec{f}\times\vec{u} = -\nabla_H \phi$

$$\frac{\partial \phi}{\partial z} = b$$

$$\rho = \rho_0 + \delta\rho$$

$$\phi = \frac{\delta p}{\rho_0}$$

$$b = -g\frac{\rho - \rho_0}{\rho_0}$$

~~Thermodynamic~~:

Equation of state: $b = g\left[\beta_T \Delta T - \beta_S \Delta S - \beta_p \Delta p\right]$
$= g\left[\beta_T \Delta T - \beta_S \Delta S - \frac{gz}{c_s^2}\right]$ $\quad c_s^2 = \frac{\partial p}{\partial \rho}$

Thermodynamic: $\frac{Db}{Dt} = \dot{b}$, or $\frac{Db}{Dt} = 0$

- With $b = \tilde{b}(z) + b'$: $\frac{Db'}{Dt} + N^2 w = 0$ where $N^2 = \frac{\partial \tilde{b}}{\partial z}$

Dispersion relation: $\omega^2 = \frac{(k^2+l^2)N^2}{k^2+l^2+m^2}$ with rotation: $\omega^2 = \frac{N^2(k^2+l^2) + f^2 m^2}{k^2+l^2+m^2}$

Geostrophy

- $\begin{cases} f u_g = \frac{-1}{\rho_0}\frac{\partial p}{\partial y} \\ f v_g = \frac{1}{\rho_0}\frac{\partial p}{\partial x} \end{cases}$

Streamfunction: If $f = f_0$, $\rho = \rho_0$, set $\psi = \frac{p}{f_0 \rho_0}$

$$\psi_x = v_g$$
$$-\psi_y = u_g$$

- Thermal wind balance:

$$\begin{cases} f\frac{\partial v_g}{\partial z} = \frac{\partial b}{\partial x} \\ f\frac{\partial u_g}{\partial z} = -\frac{\partial b}{\partial y} \end{cases}$$

$$\Rightarrow \boxed{f\frac{\partial \vec{u}}{\partial z} = k\times\nabla b}$$

Rossby Number

$$R_o = \frac{U}{fL}$$

Stability / Buoyancy Frequency

$$N^2 = \frac{-g}{\rho_0}\left(\frac{\partial \rho_0}{\partial z}\right)_E \approx \frac{-g}{\rho_0}\left(\frac{\partial \rho_\theta}{\partial z}\right)_E = \left(\frac{\partial b_\theta}{\partial z}\right)_E .$$

Stable if $N^2 > 0$

Unstable if $N^2 < 0$

Shallow Water Equations

- Barotropic:

Continuity: $\frac{\partial h}{\partial t} + \nabla\cdot(h\vec{u}) = 0$ $\quad h = H + \eta$

Momentum: $\frac{D\vec{u}}{Dt} + \vec{f}\times\vec{u} = -g\nabla\eta$

- Waves:

$$\omega^2 = f_0^2 + gH(k^2+l^2)$$

- Reduced gravity: $\vec{u}_2 = 0, \quad h = \eta_0 - \eta_1$

Continuity: $\frac{Dh}{Dt} + h\nabla\cdot\vec{u} = 0$

Momentum: $\frac{D\vec{u}}{Dt} + \vec{f}\times\vec{u} = g'\nabla\eta_1$

- Shallow water PV:

$$\frac{D}{Dt}\left(\frac{f+\zeta}{h}\right) = 0$$

Planetary Geostrophy

- One layer: $\vec{f}\times\vec{u} = -g\nabla\eta$

$\frac{\partial h}{\partial t} + \nabla\cdot(h\vec{u}) = 0$

- Boussinesq

$\vec{f}\times\vec{u} = -\nabla\phi$

$\nabla\cdot\vec{u} = 0$

$\frac{\partial\phi}{\partial z} = b$

$\frac{Db}{Dt} = 0$

Quasigeostrophy

$\frac{Dq}{Dt_g} = 0 \qquad q = \zeta_g + \beta y - \frac{f_0}{H}\eta \qquad \frac{D}{Dt_g} := \frac{\partial}{\partial t} + u_g\cdot\nabla$

- Streamfunction $\psi = \frac{g\eta}{f_0} \qquad q = \nabla^2\psi + \beta y - \frac{1}{L_D^2}\psi \qquad L_D = \frac{\sqrt{gH}}{f_0}$

$$\frac{\partial q}{\partial t} + J[\psi, q] = 0$$

$$J[A,B] = A_x B_y - A_y B_x$$

- Dispersion relation $\omega = \frac{-\beta k}{k^2 + l^2 + L_D^{-2}}$

- Continuously stratified QG

$$\frac{Dq}{Dt_g} = 0 \qquad q = \zeta_g + \beta y + \frac{\partial}{\partial z}\left(\frac{f_0 b'}{N^2}\right)$$

$$q = \nabla^2 \psi + \beta y + f_0^2 \frac{\partial}{\partial z}\left(\frac{1}{N^2}\frac{\partial \psi}{\partial z}\right)$$

- Dispersion relation $\omega = \dfrac{-\beta k}{k^2 + l^2 + m^2 \frac{f_0^2}{N^2}}$

Viscous Stress

- $\nabla \cdot \overleftrightarrow{\Pi} = \underbrace{-\nabla p}_{\text{normal stresses}} + \underbrace{\nabla \cdot \overleftrightarrow{\tau}}_{\text{shear stresses}}$ surface stress tensor
- For Newtonian Fluids: $\overleftrightarrow{\tau} = \mu \nabla \vec{v}$
- Acceleration due to viscosity: $\frac{1}{\rho_0} \nabla \cdot \overleftrightarrow{\tau} = \nu \nabla^2 \vec{v}$
- Scaling: $\nu \nabla^2 \vec{v} \approx \nu \frac{\partial^2 \vec{v}}{\partial z^2} =: \frac{1}{\rho_0} \frac{\partial \vec{\tau}}{\partial z}$, $\vec{\tau} = (\overleftrightarrow{\tau}^{zx}, \overleftrightarrow{\tau}^{zy})$

Ekman

- $f \times \vec{u} = -\nabla_H \phi + \frac{1}{\rho_0}\frac{\partial \vec{\tau}}{\partial z}$ Frictional Geostrophic Balance
- Ekman flow $\vec{u}_{Ek} = \vec{u} - \vec{u}_g$

$$f \times \vec{u}_{Ek} = \frac{1}{\rho_0}\frac{\partial \vec{\tau}}{\partial z}$$

Ekman Number

$$Ek = \frac{\nu}{f_0 H^2}$$

Ekman Depth

$$D = \sqrt{\frac{\nu}{f_0}}$$

- Ekman transport

$$\vec{M}_{Ek} = \int_D^0 \vec{u}_{Ek}\, dz$$

$$\boxed{\vec{M}_{Ek} = \frac{1}{\rho_0 f} \vec{\tau}_s \times \hat{k}}$$

- Ekman pumping

$$w_{Ek} = \frac{1}{\rho_0} \nabla \times \left(\frac{\vec{\tau}_s}{f}\right) \cdot \hat{k} = \frac{1}{\rho_0} \operatorname{curl}\left(\frac{\vec{\tau}_s}{f}\right)$$

Barotropic Vorticity Equation

$$\frac{\partial \zeta_{BT}}{\partial t} + \text{curl}\left[\int_H^\eta \vec{v}\cdot\nabla\vec{u}\, dz\right] + \beta V_{BT}$$
$$= -\text{curl}\left[\phi_B \nabla H\right] + \text{curl}\left[\int_H^\eta \nu \nabla^2 \vec{u}\, dz\right] + \frac{1}{\rho_0}\text{curl}\,\vec{\tau}$$

- Sverdrup balance $\boxed{\beta V_{BT} = \frac{1}{\rho_0} \nabla \times \vec{\tau} \cdot \hat{k}}$