$\boxed{\text{Boussinesq Equations}}$

$$\nabla \cdot \vec{v} = 0 \qquad \phi = \frac{\delta p}{\rho_0}$$

$$\begin{cases} \dfrac{D\vec{u}}{Dt} + f \times \vec{u} = -\nabla \phi \\ \left(\dfrac{Dw}{Dt}\right) + \dfrac{\partial \phi}{\partial z} = b \end{cases} \qquad b = -g \frac{\delta \rho}{\rho_0}$$

$$\frac{D}{Dt}\left[\rho - \frac{p}{c_s^2}\right] = -\frac{\dot{Q}\rho_0 \beta_T}{c_p}$$

Derivation of momentum equation

- Assume $\dfrac{\delta \rho}{\rho_0} \ll 1$
- Write $\rho = \rho_0 + \delta\rho(x,y,z,t)$
  - Associate ρ_0 with hydrostatically balanced reference p_0:

$$\frac{dp_0}{dz} = -\rho_0 g$$

$$p = p_0(z) + \delta p(x,y,z,t)$$

- Plug in stuff into momentum equations:

$$(\rho_0 + \delta\rho)\left(\frac{D\vec{v}}{Dt} + 2\Omega \times \vec{v}\right) = -\nabla \delta p - \frac{\partial p_0}{\partial z}\hat{k} - (\rho_0 + \delta\rho)g\hat{k}$$

$$\Big(1 + \underbrace{\cancel{\frac{\delta\rho}{\rho_0}}}_{\approx 0}\Big)\left(\frac{D\vec{v}}{Dt} + 2\Omega \times \vec{v}\right) = -\nabla \frac{\delta p}{\rho_0} - \frac{\delta\rho}{\rho_0} g \hat{k}$$

$$\Rightarrow \frac{D\vec{v}}{Dt} + 2\Omega \times \vec{v} = -\nabla \phi + b\hat{k}$$

Internal gravity waves

- Linearized Boussinesq equations:

$$\frac{\partial u'}{\partial t} = -\nabla \phi'$$

$$\frac{\partial w'}{\partial t} = -\frac{\partial \phi'}{\partial z} + b'$$

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0$$

$$\frac{\partial b'}{\partial t} + w' N^2 = 0$$

Thermodynamic equation

$$\frac{D}{Dt} b = 0 \qquad b = \tilde{b}(z) + b'$$

$$\frac{Db'}{Dt} + w \frac{\partial \tilde{b}}{\partial z} = 0$$

$$\Rightarrow \frac{Db'}{Dt} + N^2 w = 0 .$$

$$\rightsquigarrow \left[\frac{\partial^2}{\partial t^2} \left(\nabla^2 + \frac{\partial^2}{\partial z^2} \right) + N^2 \nabla^2 \right] w' = 0$$

Dispersion relation

$$\omega^2 = \frac{(k^2 + l^2) N^2}{(k^2 + l^2 + m^2)}$$

Buoyancy Frequency

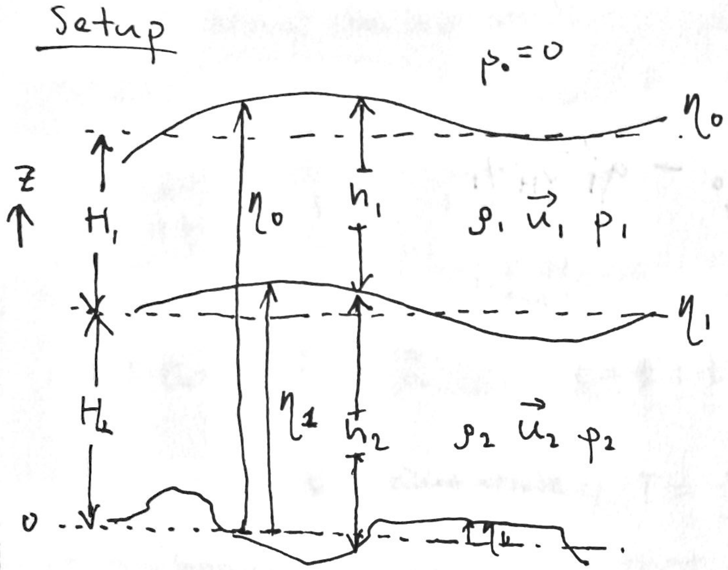
$\cancel{N^2}$

$$\frac{\partial^2 \delta z}{\partial t^2} = \frac{g}{\rho_0} \left(\frac{\partial \rho_0}{\partial z} \right) \delta z = -N^2 \delta z$$

$$N^2 = -\frac{g}{\rho_0} \left(\frac{\partial \rho_0}{\partial z} \right) = \frac{\partial b_0}{\partial z} .$$

Two-layer Shallow Water

Setup



Assumptions

- No surface pressure
- Hydrostatic balance yields pressure gradients in each layer.
- $\rho_2 > \rho_1$
- $\left|\dfrac{\rho_2 - \rho_1}{\rho_1}\right| \ll 1$

- Compute pressure in each layer.
  - In first layer at height z:

$$\int_z^{\eta_0} \frac{\partial p_1}{\partial z}\, dz = -\int_z^{\eta_0} \rho_1 g \, dz$$

$$\cancel{p_1(\eta_0)} - p_1(z) = -\rho_1 g(\eta_0 - z)$$

$$\Rightarrow \quad p_1(z) = -\rho_1 g(z - \eta_0) \quad \left(\Rightarrow \frac{1}{\rho_1}\nabla_H p_1 = g\nabla_H \eta_0\right)$$

  - second layer:

$$\int_z^{\eta_0} \frac{\partial p}{\partial z}\, dz = \left[\int_z^{\eta_1} \frac{\partial p_2}{\partial z}\, dz + \int_{\eta_1}^{\eta_0} \frac{\partial p_1}{\partial z}\, dz\right]$$

$$+ p(z) = \cancel{p_2(\eta_1) - p_2(z) + p_1(\eta_0) - p_1(\eta_1)}$$

$$+ p(z) = \int_z^{\eta_1} \rho_2 g \, dz + \int_{\eta_1}^{\eta_0} \rho_1 g \, dz$$

$$\Rightarrow \quad p_2(z) = \rho_2 g(\eta_1 - z) + \rho_1 g(\eta_0 - \eta_1)$$

$$= \rho_1 g \eta_0 + (\rho_2 - \rho_1) g \eta_1 - \rho_2 g z$$

$$= \rho_1 g \eta_0 + \rho_1 g_1' \eta_1 - \rho_2 g z$$

$$\Rightarrow \quad \nabla_H p_2 = \rho_1 g \nabla_H \eta_0 + \rho_1 g_1' \nabla_H \eta_1$$

• Use pressure to compute horizontal pressure gradients / momentum eq:

$$\frac{D\vec{u}_1}{Dt} + \vec{f} \times \vec{u}_1 = -\frac{1}{\rho_1}\nabla_H p_1 = -g\nabla_H \eta_0$$

$$\frac{D\vec{u}_2}{Dt} + f \times \vec{u}_2 = -\frac{1}{\rho_2}\nabla_H p_2 = -g\nabla_H \eta_0 - g_1'\nabla_H \eta_1$$

• Continuity equation:

$$\frac{\partial h_1}{\partial t} + \nabla\cdot(\vec{u}_1 h_1) = 0$$

$$\frac{\partial h_2}{\partial t} + \nabla\cdot(u_2 h_2) = 0$$

In general, for N layers:

$$p_n = \rho_1 \sum_{i=1}^{n-1} g_i' \eta_i \qquad g_i' = \frac{\rho_{i+1} - \rho_i}{\rho_1} g$$

$$\eta_n = \eta_b + \sum_{i=n+1}^{i=N} h_i$$

use boussinesq approx $\rho_i \approx \rho_1$

Momentum equations:

$$\frac{D\vec{u}_n}{Dt} + \vec{f} \times \vec{u}_n = -\frac{1}{\rho_n}\nabla p_n = -\sum_{i=1}^{n-1} g_i' \nabla \eta_i - g\nabla_H \eta_0$$

Mass conservation:

$$\frac{\partial h_n}{\partial t} + \nabla\cdot(h_n \vec{u}_n) = 0.$$

Dispersion relationship

$$\omega^2 = f_0^2 + gH\left(\frac{\rho_2 - \rho_1}{\rho_2 + \rho_1}\right)(k^2 + l^2),$$ I think. Total guess though.

$\boxed{\text{Quasigeostrophic Equations}}$

- Start from shallow water:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - (f_0 + \beta y)v = -g\frac{\partial \eta}{\partial x}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + (f_0 + \beta y)u = -g\frac{\partial \eta}{\partial y}$$

- Nondimensionalization:

$$u = U\hat{u} \qquad t = T\hat{t} \qquad (x,y) = L(\hat{x},\hat{y})$$

- Advective timescale: $T = \frac{L}{U}$
- Choose $\beta = \frac{U}{L^2}\hat{\beta}$ so that $\frac{\beta y}{f_0} = \frac{U}{fL}\hat{\beta}\hat{y} = Ro\,\hat{\beta}\hat{y}$.
- Geostrophic scaling for sea level:

$$f_0 U \sim \frac{\eta}{L} g$$

$$\eta \sim \frac{f_0 U L}{g}\hat{\eta}$$

- Momentum equations:

$$\varepsilon\frac{D\hat{u}}{D\hat{t}} = -\frac{\partial\hat{\eta}}{\partial\hat{x}} + (1+\varepsilon\hat{\beta}\hat{y})\hat{v}$$

$$\varepsilon\frac{D\hat{v}}{D\hat{t}} = -\frac{\partial\hat{\eta}}{\partial\hat{y}} - (1+\varepsilon\hat{\beta}\hat{y})\hat{u} \qquad \varepsilon = Ro = \frac{U}{fL}$$

- Continuity equation:

$$\varepsilon\frac{D\hat{\eta}}{D\hat{t}} = \left(\frac{L_D^2}{L^2} + \varepsilon\hat{\eta}\right)\left(\frac{\partial\hat{u}}{\partial\hat{x}} + \frac{\partial\hat{v}}{\partial\hat{y}}\right) \qquad \boxed{\text{For QG: } L_D \approx L}$$

- Asymptotic expansion:

$$\hat{u} = \hat{u}_0 + \varepsilon\hat{u}_1 + \mathcal{O}(\varepsilon^2)$$

$\hat{u}_0$: geostrophic velocity (show in a bit)

- Geostrophic advection:

$$\frac{D}{D\tilde{t}} = \frac{\partial}{\partial\tilde{t}} + \vec{u}_0\cdot\nabla + \varepsilon\vec{u}_1\cdot\nabla + \mathcal{O}(\varepsilon^2)$$

$$= \frac{D}{D\tilde{t}_0} + \mathcal{O}(\varepsilon).$$

- Rewrite momentum equations using geostrophic advection:

$$\varepsilon \frac{D\hat{u}}{D\hat{t}} = -\frac{\partial \hat{\eta}}{\partial \hat{x}} + (1+\varepsilon \hat{\beta}\hat{y})\hat{v}$$

$$\varepsilon \frac{D\hat{v}}{D\hat{t}} = -\frac{\partial \hat{\eta}}{\partial \hat{y}} - (1+\varepsilon \hat{\beta}\hat{y})\hat{u}$$

$$\rightsquigarrow \quad \varepsilon \frac{D\tilde{u}}{D\tilde{t}_0} + \mathcal{O}(\varepsilon^2) = -\frac{\partial \hat{\eta}}{\partial \tilde{x}} + \hat{v}_0 + \varepsilon \tilde{v}_1 + \varepsilon \hat{\beta}\hat{y}\tilde{v}_0$$

$$\varepsilon \frac{D\tilde{v}}{Dt_0} + \mathcal{O}(\varepsilon^2) = -\frac{\partial \tilde{\eta}}{\partial \tilde{y}} - \hat{u}_0 - \varepsilon \tilde{u}_1 - \varepsilon \hat{\beta}\hat{y}\tilde{u}_0$$

- Collect $\mathcal{O}(1)$ terms and $\mathcal{O}(\varepsilon)$ terms:

$$\mathcal{O}(1): \begin{cases} \tilde{v}_0 = \dfrac{\partial \hat{\eta}}{\partial \tilde{x}} \\ u_0 = -\dfrac{\partial \tilde{\eta}}{\partial \tilde{y}} \end{cases} \quad \underline{\underline{\text{GSB}}}$$

$$\frac{\partial \tilde{u}_0}{\partial \tilde{x}} + \frac{\partial \tilde{v}_0}{\partial \tilde{y}} = 0 \quad \text{Horizontal Nondivergence}$$

$$\mathcal{O}(\varepsilon): \quad \frac{D\tilde{u}}{D\tilde{t}_0} = \hat{\beta}\hat{y}\tilde{v}_0 + \tilde{v}_1 .$$

$$\frac{D\tilde{v}}{D\tilde{t}_0} = -\hat{\beta}\hat{y}\tilde{u}_0 - \tilde{u}_1$$

$$\frac{D\hat{\eta}}{D\tilde{t}} = -\left(\frac{L_D}{L}\right)^2 \left(\frac{\partial \tilde{u}_1}{\partial \tilde{x}} + \frac{\partial \tilde{v}_1}{\partial \tilde{y}}\right).$$

- Derive vorticity equation (curl of momentum)
  - Define $\tilde{\zeta}_0 = \dfrac{\partial \tilde{v}_0}{\partial \tilde{x}} - \dfrac{\partial \tilde{u}_0}{\partial \tilde{y}}$

$$\Rightarrow \quad \frac{D\tilde{\zeta}_0}{Dt_0} = -\frac{\partial}{\partial \tilde{x}}\left(\hat{\beta}\hat{y}\tilde{u}_0\right) - \frac{\partial}{\partial \tilde{y}}\left(\hat{\beta}\hat{y}\tilde{v}_0\right) - \left(\frac{\partial \tilde{u}_1}{\partial \tilde{x}} + \frac{\partial \tilde{v}_1}{\partial \tilde{y}}\right)$$

$$= -\underline{\tilde{u}_0} \cdot \nabla(\hat{\beta}\hat{y}) - \hat{\beta}\hat{y}\left(\frac{\partial \tilde{u}_0}{\partial \tilde{x}} + \frac{\partial \tilde{v}_0}{\partial \tilde{y}}\right) - \left(\frac{\partial \tilde{u}_1}{\partial \tilde{x}} + \frac{\partial \tilde{v}_1}{\partial \tilde{y}}\right)$$

$$\Rightarrow \quad \frac{D}{D\tilde{t}_0}\left(\tilde{\zeta}_0 + \hat{\beta}\hat{y}\right) = \left(\frac{L}{L_D}\right)^2 \frac{D\tilde{\eta}}{D\tilde{t}}$$

$$\Rightarrow \quad \frac{D}{D\tilde{t}_0}\left[\zeta_0 + \hat{\beta}\hat{y} - \left(\frac{L}{L_D}\right)^2 \tilde{\eta}\right] = 0$$

- Redimensionalize:

$$\frac{D}{D\tilde{t}_0} = \frac{U}{L}\frac{D}{Dt_g}$$

$$\hat{\zeta}_0 \sim \frac{\partial \tilde{v}}{\partial \tilde{x}} \sim \frac{L}{U}\frac{\partial v}{\partial x} \sim \frac{L}{U}\zeta$$

$$\hat{\beta}\hat{y} = \frac{L}{U}\beta y$$

$$\left(\frac{L}{L_D}\right)^2 \tilde{\eta} = \frac{L^3 f_0^2}{gH} \cdot \frac{g}{f_0 U L}\eta = \frac{L}{U}\frac{f_0}{H}\eta$$

$$\Rightarrow \boxed{\frac{Dq}{Dt_g} = 0 \quad \text{where } q = \zeta + \beta y - \frac{f_0}{H}\eta}$$ QG Potential Vorticity

- Define geostrophic streamfunction:

$$\psi = \frac{g\eta}{f_0}$$

$$\Rightarrow \psi_x = v_g$$

$$\psi_y = -u_g$$

$$\zeta = \nabla^2\psi$$

$$\boxed{q = \nabla^2\psi + \beta y - \frac{1}{L_D^2}\psi}$$

QG PV – streamfunction version

$$\Rightarrow \frac{\partial q}{\partial t} + u_g\frac{\partial q}{\partial x} + v_g\frac{\partial q}{\partial y} = 0$$

$$\Rightarrow \boxed{\frac{\partial q}{\partial t} + J[\psi, q] = 0}$$

$$J[A,B] = A_x B_y - A_y B_x$$

Dispersion relation (Rossby waves)

- Linearization:

$$\frac{\partial}{\partial t}\left(\nabla^2\psi - \frac{1}{L_D^2}\psi\right) + \beta\frac{\partial\psi}{\partial x} = 0$$

- Dispersion relation:

$$\omega = \frac{-\beta k}{k^2 + l^2 + k_D^2}$$

- Group velocity

$$c_g^{(x)} = \frac{\beta(k^2 - l^2 - k_D^2)}{k^2 + l^2 + k_D^2}$$

  - For long waves: $k \ll k_D$
    westward group velocity.

$\boxed{\text{The Ekman Layer}}$

Viscous Stress

- Acceleration due to viscosity: $\frac{1}{\rho_0} \nabla \cdot \overleftrightarrow{\tau}_{3D}$ $\overleftrightarrow{\tau}_{3D}$ - stress tensor
- Newtonian fluids: $\overleftrightarrow{\tau}_{3D} = \mu \nabla \vec{v}$
- Constant viscosity:

$$\frac{1}{\rho_0} \nabla \cdot \overleftrightarrow{\tau}_{3D} = \frac{1}{\rho_0} \nabla \cdot (\mu \nabla \vec{v}) = \frac{\mu}{\rho_0} \nabla^2 \vec{v} = \nu \nabla^2 \vec{v}$$

molecular Viscosity.

Eddy viscosity

- Take the x-momentum equation with no x, y dependence:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = fv + \nu \nabla^2 u$$

$$\frac{\partial u}{\partial t} + \frac{\partial (uu)}{\partial x} - \cancel{u\frac{\partial u}{\partial x}} + \frac{\partial (uv)}{\partial y} - \cancel{u\frac{\partial v}{\partial y}} + \frac{\partial (uw)}{\partial z} - \cancel{u\frac{\partial w}{\partial z}} = fv + \nu \nabla^2 u$$

$$(\nabla \cdot \vec{v} = 0)$$

$$\frac{\partial u}{\partial t} + \cancel{\frac{\partial}{\partial x}}(uu) + \cancel{\frac{\partial (uv)}{\partial y}} + \frac{\partial (uw)}{\partial z} = fv + \nu \nabla^2 u$$

no x-dependence   no y-dependence

$$\sim \quad \frac{\partial u}{\partial t} - \frac{\partial (uw)}{\partial z} = fv + \nu \nabla^2 u$$

- Reynold's decomposition:

$$\frac{\partial \bar{u}}{\partial t} + \frac{\partial u'}{\partial t} - \frac{\partial}{\partial z}\left((\bar{u}+u')(\bar{w}+w')\right) = f\bar{v} + fv' + \nu \nabla^2(\bar{u}+u')$$

- Averaging:

$$\frac{\partial \bar{u}}{\partial t} - \frac{\partial \bar{u}\bar{w}}{\partial z} = f\bar{v} + \nu \nabla^2 \bar{u} + \frac{\partial}{\partial z}(\overline{u'w'}).$$

- $\frac{\partial}{\partial z}(\overline{u'w'})$ is a small-scale turbulence term — "Reynold's stresses"
- People represent this small-scale turbulence as a viscosity:

$$\frac{\partial}{\partial z}\left(\overline{u'w'}\right) \approx \nu_{eff} \frac{\partial^2 \bar{u}}{\partial z^2}$$

Effective viscosity / eddy viscosity.

- Typically, $\nu_{eff} >> \nu_{mol}$:

$$\nu_{eff} \sim 10^{-1} \frac{m^2}{s} \qquad \nu \sim 10^{-6} \frac{m^2}{s}$$

- Dropping bars for convenience, the equations for large-scale flow are:

$$\begin{cases} \dfrac{Du}{Dt} - fv = \nu_{eff} \dfrac{\partial^2 u}{\partial z^2} \\[2mm] \dfrac{Dv}{Dt} + fu = \nu_{eff} \dfrac{\partial^2 v}{\partial z^2} \end{cases}$$

$$\Rightarrow \quad \boxed{\frac{D\vec{u}}{Dt} + \vec{f} \times \vec{u} = \nu_{eff} \frac{\partial^2 \vec{u}}{\partial z^2} \approx \frac{1}{\rho_0} \left(\frac{\partial \vec{\tau}}{\partial z} \right)}$$

where $\vec{\tau} \approx \cancel{\left(\tau^{xz}_{3D}, \tau^{yz}_{3D} \right)} \approx \left(\tau^{(x)}, \tau^{(y)} \right)$

$\left(\tau^{zx}_{3D}, \tau^{zy}_{3D} \right)$

Ekman flow

- At the surface, shear is induced by wind stress

$$\vec{\tau}(z=0) = \left(\tau^{(x)}(0), \tau^{(y)}(0) \right) = \left(\tau_{atm,x}, \tau_{atm,y} \right).$$

- Boussinesq equations:

$$\begin{cases} \dfrac{D\vec{u}}{Dt} + \vec{f} \times \vec{u} = -\nabla_H \phi + \dfrac{1}{\rho_0} \dfrac{\partial \vec{\tau}}{\partial z} \\[2mm] \dfrac{\partial \phi}{\partial z} = b \end{cases}$$

- Small Rossbly number: $R_o \ll 1$

$$\boxed{\vec{f} \times \vec{u} = -\nabla_H \phi + \frac{1}{\rho_0} \frac{\partial \vec{\tau}}{\partial z}}$$

frictional-geostrophic balance.

- The Ekman number:

$$\underbrace{\vec{f} \times \vec{u}}_{\sim f_0 U} = \underbrace{-\nabla_H \phi}_{\sim \Phi / L} + \underbrace{\frac{1}{\rho_0} \frac{\partial \vec{\tau}}{\partial z}}_{\sim \nu \frac{U}{H^2}}$$

- Assume $\Phi \sim f_0 U L$ (geostrophic balance)
- Ratio of viscous terms to Coriolis terms:

$$Ek = \frac{\nu}{f_0 H^2}$$

- Frictional effects become important in upper layer where $H^2 < \frac{\nu}{f_0}$

- Ekman flow:

write $\vec{u} = \vec{u}_g + \vec{u}_{Ek}$ $\qquad \vec{u}_{Ek}$: Ekman component

$\phi = \phi_g + \phi_{Ek}$

$\cancel{f \times (\vec{u} - \vec{u}_g)}$

$$f \times \vec{u} = -\nabla_H \phi + \nu \frac{\partial^2 \vec{u}}{\partial z^2}$$

- Geostrophic component: $f \times u_g = -\nabla_H \phi_g$

$$\rightsquigarrow \quad f \times \vec{u}_{Ek} = -\nabla_H \phi_{Ek} + \nu \frac{\partial^2 \vec{u}_{Ek}}{\partial z^2}.$$

- Integrate over Ekman layer:

$$\int_\delta^0 \vec{f} \times \vec{u}_{Ek}\, dz = \int_\delta^0 \nu \frac{\partial}{\partial z}\left(\frac{\partial u_{Ek}}{\partial z}\right) dz$$

$$f \times \underbrace{\int_\delta^0 \vec{u}_{Ek}\, dz}_{=\vec{M}_{Ek}} = \nu \left[\underbrace{\cancel{\frac{\partial u_{Ek}}{\partial z}(D)}}_{=0} - \underbrace{\frac{\partial u_{Ek}}{\partial z}(0)}_{\nu \frac{\partial u_{Ek}}{\partial z} = \frac{\vec{\tau}}{\rho_0}} \right]$$

$$\vec{f} \times \vec{M}_{Ek} = \cancel{0} + \frac{\vec{\tau}}{\rho_0}$$

- Take ~~curl~~ ~~and~~ $\hat{k} \times (\cdot)$:

$$\boxed{\vec{M}_{Ek} = \frac{1}{\rho_0 f} \vec{\tau} \times \hat{k}} = \left(\frac{\tau^{(y)}}{\rho_0 f}, -\frac{\tau^{(x)}}{\rho_0 f} \right)$$

- Observations:

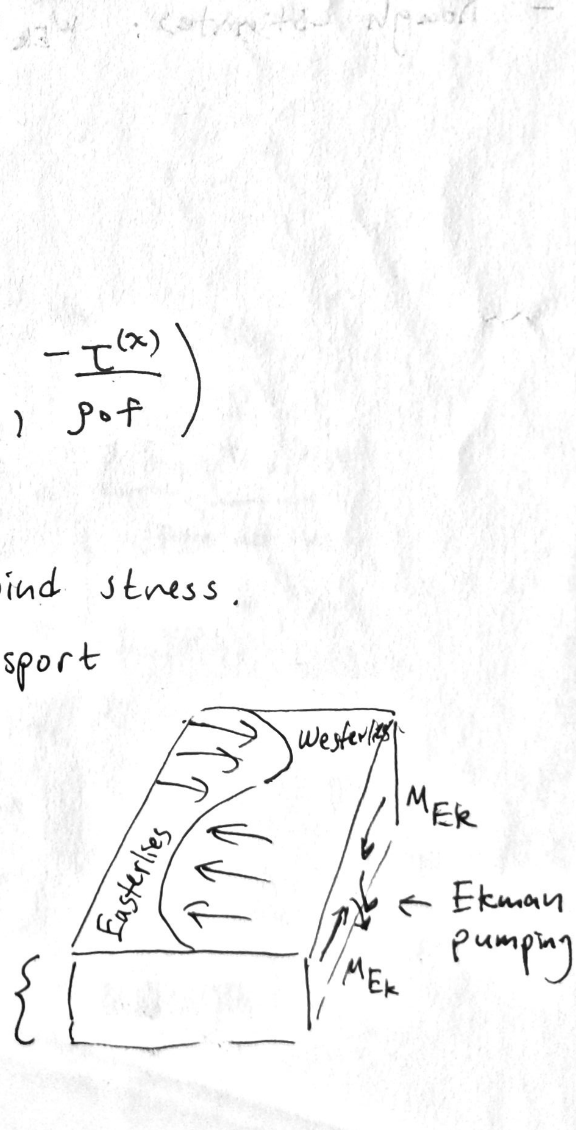
1) Ekman transport is perpendicular to wind stress.

Westerly winds: ~~M_{Ek}~~ southward transport

Easterly winds: northward transport

2) Ekman pumping:

* Mass convergence induces downwelling



- Ekman pumping:
  - Want $w(D)$ vertical velocity at base of Ekman layer

$$-\int_D^0 \frac{\partial w}{\partial z}\, dz = \int_D^0 \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\, dz \quad \text{(continuity equation)}$$

$$\underbrace{w(D) - \cancel{w(0)}}_{\substack{\text{assumed} \\ \text{small} \\ \text{(on average}}} = \frac{\partial}{\partial x}\underbrace{\int_D^0 u\, dz}_{M_{Ek}^{(x)}} + \frac{\partial}{\partial y}\underbrace{\int_D^0 v\, dz}_{M_{Ek}^{(y)}}$$

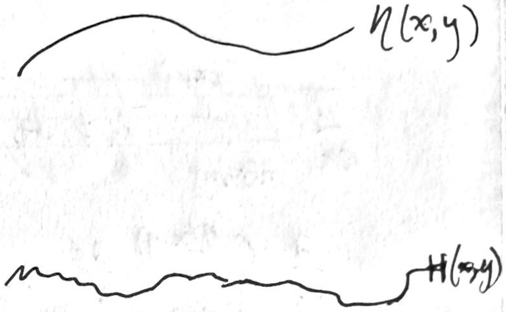
$$= \frac{1}{\rho_0}\left[\frac{\partial}{\partial x}\left(\frac{\tau_y}{f}\right) - \frac{\partial}{\partial y}\left(\frac{\tau_x}{f}\right)\right]$$

$$\boxed{w(D) = \frac{1}{\rho_0}\nabla \times \left(\frac{\tau}{f}\right)\cdot \hat{k}} \quad \text{Ekman pumping.}$$

- Rough estimates: $w_{Ek}(D) \sim 1\text{m}/5 \text{ days}$.

The Barotropic Vorticity Equation

- Integrate momentum equations from floor to top:



$$\frac{\partial u}{\partial t} + (\vec{v}\cdot\nabla)u - fv = -\frac{\partial \phi}{\partial x} + \nabla_h\cdot(\nu_h \nabla_h u) + \frac{\partial}{\partial z}\left(\nu_z \frac{\partial u}{\partial z}\right)$$

$$\frac{\partial v}{\partial t} + (\vec{v}\cdot\nabla)v + fu = -\frac{\partial \phi}{\partial y} + \nabla_h\cdot(\nu_h \nabla_h v) + \frac{\partial}{\partial z}\left(\nu_z \frac{\partial v}{\partial z}\right)$$

$$u_{BT} = \int_H^\eta u\, dz \qquad v_{BT} = \int_H^\eta v\, dz$$

$$\Rightarrow (1)\quad \frac{\partial u_{BT}}{\partial t} + \int_H^\eta (\vec{v}\cdot\nabla)u\, dz - f v_{BT} = -\int_H^\eta \frac{\partial \phi}{\partial x}\, dz + \int_H^\eta \nabla_h\cdot(\nu_H \nabla_h u)\, dz + \int_H^\eta \frac{\partial}{\partial z}\left(\nu_z \frac{\partial u}{\partial z}\right) dz$$

$$(2)\quad \frac{\partial v_{BT}}{\partial t} + \int_H^\eta (\vec{v}\cdot\nabla)v\, dz + f u_{BT} = -\int_H^\eta \frac{\partial \phi}{\partial y}\, dz + \int_H^\eta \nabla_h\cdot(\nu_h \nabla_h v)\, dz + \int_H^\eta \frac{\partial}{\partial z}\left(\nu_z \frac{\partial v}{\partial z}\right) dz$$

- Note that $\displaystyle\int_H^\eta \frac{\partial}{\partial z}\left(\nu_z \frac{\partial u}{\partial z}\right) dz = \nu_z \frac{\partial u}{\partial z}\Big|_{z=\eta} - \nu_z \frac{\partial u}{\partial z}\Big|_{z=H}$

Assume no bottom friction for simplicity:

$$= \nu_z \frac{\partial u}{\partial z}\Big|_{z=\eta} = \frac{1}{\rho_0}\tau_x$$

- Take curl: $\frac{\partial}{\partial x}(2) - \frac{\partial}{\partial y}(1)$. Define $\zeta_{BT} = \cancel{v_{BT}}\,\frac{\partial}{\partial x} v_{BT} - \frac{\partial}{\partial y} u_{BT}$

* Note that $\displaystyle\frac{\partial}{\partial x}(f u_{BT}) - \frac{\partial}{\partial y}(f v_{BT}) = \cancel{f}\, f\frac{\partial u_{BT}}{\partial x} + \underbrace{f\frac{\partial v_{BT}}{\partial y} + \beta v_{BT}}_{\text{product rule}}$

$$= f\underbrace{\left(\frac{\partial u_{BT}}{\partial x} + \frac{\partial v_{BT}}{\partial y}\right)}_{= \int \frac{\partial w}{\partial z} dz \approx 0} + \beta v_{BT}.$$

$$\Rightarrow \frac{\partial \zeta_{BT}}{\partial t} + \text{curl}\left[\int_H^\eta (\vec{v}\cdot\nabla)\vec{u}\, dz\right] + \beta v_{BT} = -\text{curl}\left[\int_H^\eta \nabla_h \phi\, dz\right]$$

$$+ \text{curl}\left[\int_H^\eta \nabla_h\cdot(\nu_h \nabla_h \vec{u})\, dz\right] + \frac{1}{\rho_0}\nabla\times\vec{\tau}.$$

* Use Leibniz rule on curl$\left[\int_{H(x,y)}^{\eta(x,y)} \nabla_h \phi \, dz\right]$

$$\cancel{\frac{\partial}{\partial x}\int_{H(x,y)}^{\eta(x,y)} \nabla_h \phi \, dz = \int_{H(x,y)}^{\eta(x,y)} \frac{\partial}{\partial x}(\nabla_h \phi)\, dz} =$$

$$\frac{\partial}{\partial x}\int_{H(x,y)}^{\eta(x,y)} \frac{\partial \phi}{\partial y}\, dz = \int_{H(x,y)}^{\eta(x,y)} \frac{\partial \phi}{\partial x \partial y}\, dz + \left.\frac{\partial \eta}{\partial x}\frac{\partial \phi}{\partial y}\right|_{z=\eta} - \left.\frac{\partial H}{\partial x}\frac{\partial \phi}{\partial y}\right|_{z=H}$$

$$\rightsquigarrow \text{curl}\left[\int_{H(x,y)}^{\eta(x,y)} \nabla_h \phi\, dz\right] \approx \left.\frac{\partial H}{\partial y}\frac{\partial \phi}{\partial x}\right|_{z=H} - \left.\frac{\partial H}{\partial x}\frac{\partial \phi}{\partial y}\right|_{z=H} \quad \text{assuming } \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y} \text{ small.}$$

Better: use Leibniz integral rule backwards on

$$\nabla_h \int_{H(x,y)}^{\eta(x,y)} \phi\, dz = \int_H^\eta \nabla_h \phi\, dz - \phi_B \nabla_h H \qquad \phi_B = \phi|_{z=H} \text{ bottom pressure}$$

(again assuming $\nabla_h \phi|_{z=\eta}$ small). Then curl $\nabla_h\left(\int_H^\eta \phi\, dz\right) = 0$.

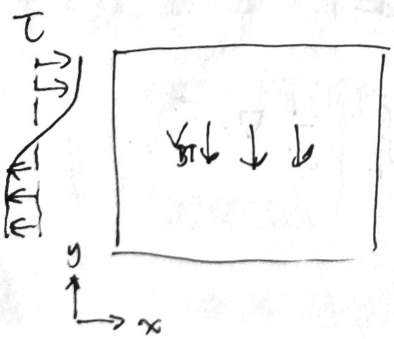
$$\rightsquigarrow \boxed{\begin{aligned} \frac{\partial \zeta_{BT}}{\partial t} + \text{curl}\left[\int_H^\eta (\vec{v}\cdot\nabla)\vec{u}\, dz\right] + \beta V_{BT} &= -\text{curl}\left[\phi_B \nabla H\right] \\ &+ \text{curl}\left[\int_H^\eta \nabla_h\cdot(\nu_h \nabla_h \vec{u})\, dz\right] + \frac{1}{\rho_0}\nabla\times\vec{\tau} \\ &\text{The Barotropic vorticity equation.}\end{aligned}}$$

Sverdrup balance

- Assume steady flow, nonlinear terms are small, changes in bottom pressure are small, and viscosity is unimportant.*

$$\rightsquigarrow \boxed{\beta V_{BT} = \frac{1}{\rho_0}\nabla\times\vec{\tau}} \quad \text{Sverdrup Balance}$$

* away from boundaries.



Stommel Model for Western Boundary Currents

Equations

- Frictional GSB:

$$-fv = -\frac{\partial \phi}{\partial x} + \frac{\partial \tau^{(x)}}{\partial z}$$

$$fu = -\frac{\partial \phi}{\partial y} + \frac{\partial \tau^{(y)}}{\partial z}$$

- Take curl; integrate over ocean depth:

$$\int_H^0 f\,\nabla\cdot\vec{u}\,dz + \beta\int_H^0 v\,dz = \mathrm{curl}\left(\vec{\tau}_s - \vec{\tau}_B\right)$$

- Bottom drag: $\mathrm{curl}\,\vec{\tau}_B = r\zeta$
- $\int_H^0 f\,\nabla\cdot\vec{u}\,dz = \int_H^0 -f\frac{\partial w}{\partial z}\,dz = 0$

$$\Rightarrow \quad r\zeta + \beta V = \mathrm{curl}\,\vec{\tau}_s$$

- Streamfunction $V = \psi_x \quad U = -\psi_y, \; \zeta = \nabla^2\psi$

$$\Rightarrow \quad \boxed{r\nabla^2\psi + \beta\psi_x = \mathrm{curl}\,\vec{\tau}_s}$$

- Purely zonal wind stress is convenient

$$\vec{\tau}_s = -\cos\left(\frac{\pi y}{L_y}\right)\hat{\imath} \quad 0 \le y \le L_y$$

Scaling

- Friction becomes important only in a small boundary layer by the coast
- $|r\zeta| \sim |\beta v|$ if $r\frac{U}{L_B} \sim \beta U \Rightarrow L_B \sim \frac{r}{\beta}$
- Define Stommel number $\varepsilon_s = \frac{L_B}{L} = \frac{r}{\beta L} \ll 1$.
- Nondimensionalize

$$(x,y) = L(\hat{x},\hat{y}) \qquad \vec{\tau} = \tau_0\hat{\tau} \qquad \psi = \Psi\hat{\psi}$$

Ψ scales like $\frac{\beta\Psi}{L} = \frac{\tau_0}{L} \Rightarrow \Psi = \frac{\tau_0}{\beta}$

$$\Rightarrow \quad \frac{\partial\hat{\psi}}{\partial\hat{x}} + \varepsilon_s\hat{\nabla}^2\hat{\psi} = \mathrm{curl}\,\hat{\tau}_s$$

• Decomposition into boundary and interior:

Write $\psi = \psi_I + \psi_B$

where $\psi_I = \psi_I(x,y)$

and $\psi_B = \psi_B(\xi, y)$ where $\xi = \frac{x}{\varepsilon}$

Then $\frac{\partial \psi_B}{\partial x} = \frac{1}{\varepsilon_s}\frac{\partial \psi_B}{\partial \xi}$, $\nabla^2 \psi_B = \frac{1}{\varepsilon_s^2}\frac{\partial^2 \psi_B}{\partial \xi^2}$

$\rightsquigarrow$ $\varepsilon_s\left(\nabla^2 \psi\right) + \frac{\partial \psi}{\partial x} = \text{curl}\,\hat{\tau}$ becomes

$$\varepsilon_s\left(\nabla^2\psi_I + \frac{1}{\varepsilon_s^2}\frac{\partial^2\psi_B}{\partial\xi^2} + \frac{\partial^2\psi_B}{\partial y^2}\right) + \frac{\partial\psi_I}{\partial x} + \frac{1}{\varepsilon_s}\frac{\partial\psi_B}{\partial\xi} = \text{curl}\,\hat{\tau}_s \quad (*)$$

• Ocean interior solution: Sverdrup balance

$$\frac{\partial\psi_I}{\partial x} = \text{curl}\,\hat{\tau}_s$$

$$\Rightarrow \psi_I(x,y) = \int_0^x \text{curl}\,\hat{\tau}_s\,dx' + g(y)$$

- Assume zonal wind stress $\hat{\tau}_s^{(x)} = -\cos\pi y$

$$\text{curl}\,\hat{\tau}_s = \frac{\partial\tau^{(y)}}{\partial x} - \frac{\partial\tau_s^{(x)}}{\partial y} = -\pi\sin\pi y$$

$$\Rightarrow \psi_I(x,y) = -\pi x\sin\pi y + g(y)$$

• Ocean Boundary: $O(1)$ terms, removing Sverdrup balance

$$\rightsquigarrow \frac{\partial^2\psi_B}{\partial\xi^2} + \frac{\partial\psi_B}{\partial\xi} = 0$$

$$\rightsquigarrow \psi_B(\xi,y) = A(y) + B(y)e^{-\xi}$$

• Compose full solution and solve BCs:

$$\psi = -\pi x\sin\pi y + \underbrace{A(y) + g(y)}_{C(y)} + B(y)e^{-\frac{x}{\varepsilon}}.$$

$\psi(0,y) = 0 \Rightarrow C(y) + B(y) = 0$

$\psi(1,y) = 0 \Rightarrow C(y) - \pi\sin\pi y = 0$ $\rightsquigarrow$ Full solution.

Munk Problem

- Use lateral viscosity instead of bottom drag:

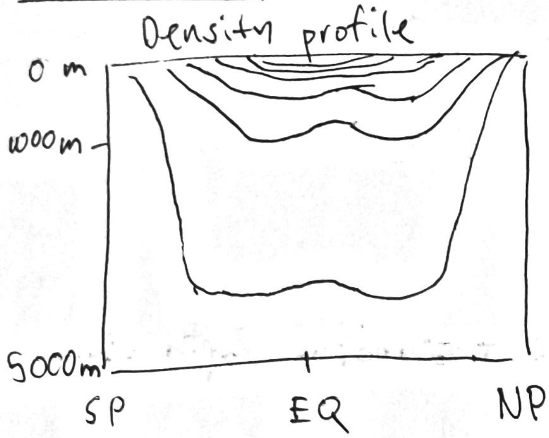
$$\beta \frac{\partial \psi}{\partial x} + \nu \nabla^2 \zeta = \text{curl } \tau_s$$

$$\Rightarrow \nu \nabla^4 \psi + \beta \frac{\partial \psi}{\partial x} = \text{curl } \tau_s .$$

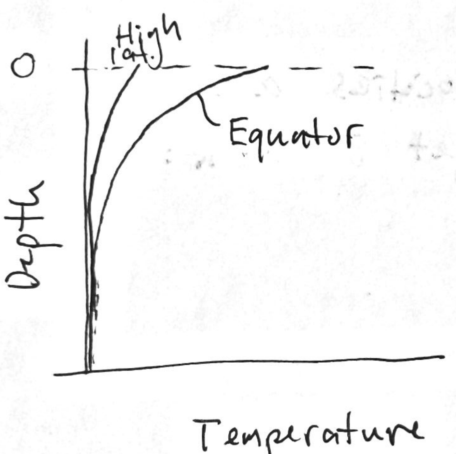
- Boundary layer $L_M = \left(\frac{\nu}{\beta}\right)^{1/3}$

Thermocline Theory

Observations



- Gradients in potential density are largest in the upper ~1 km of ocean, little change in density below this.
- shallower in equatorial zone than in midlatitudes
- Isopycnals outcrop in the high latitudes.

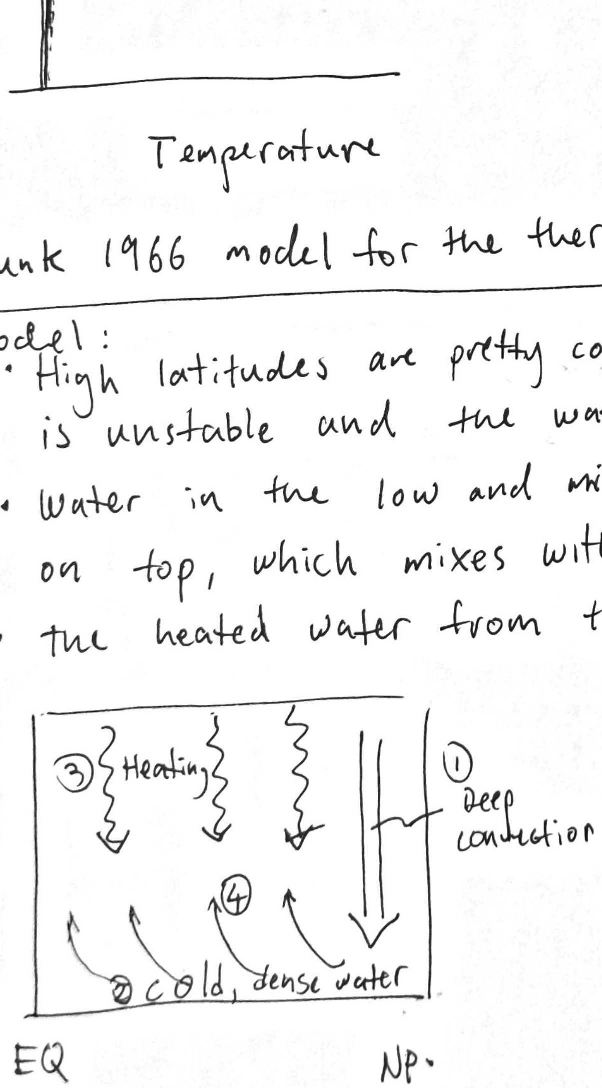


- Temperature decays ~exponentially with depth
- sharper temperature stratification for low-latitudes.

Munk 1966 model for the thermocline structure

Model:

1. • High latitudes are pretty cold, so the column there is unstable and the water sinks, and fills the abyss. 2
3. • Water in the low and midlatitudes has warmer water on top, which mixes with the abyss to heat it
- the heated water from the abyss ~~m~~ then upwells.



In the interior and low latitude this results in an advection diffusion balance:

$$w \frac{dT}{dz} = K \frac{d^2T}{dz^2}$$

w: upwelling velocity in low/mid latitudes

K: heat diffusivity.

Advective-Diffusion equation

$$w \frac{dT}{dz} = \kappa \frac{d^2T}{dz^2}$$

Boundary conditions:

$z = 0$: $T = T_S$

$z \to -\infty$ $T = T_D$ so that $\frac{dT}{dz} \to 0$.

$$T(z) = (T_S - T_D) e^{\frac{w}{\kappa} z} + T_D \qquad \delta = \frac{\kappa}{w} \text{ thermocline depth scale}$$

Thermocline Scalings without wind

If we ignore the effects of wind, the vertical velocities are primarily the result of thermal mixing and we get $\delta \sim 100\,m$:

- Equations

GSB + HSB : $f \frac{\partial \vec{u}}{\partial z} = \hat{k} \times \nabla b$

Continuity : $\nabla \cdot \vec{v} = 0$

$\Rightarrow$ vorticity balance $f \frac{\partial w}{\partial z} = \beta v$

Thermodynamic $\frac{Db}{Dt} = \kappa \frac{\partial^2 b}{\partial z^2}$

- Dimensional analysis:

$z \sim \delta \qquad x, y \sim L \qquad w \sim W \qquad u, v \sim U$

$$\Rightarrow \delta = \left(\frac{\kappa f^2 L}{\beta \Delta b}\right)^{1/3} \qquad W = \left(\frac{\kappa^2 \beta \Delta b}{f^2 L}\right)^{1/3}$$

$$\Rightarrow W \sim 10^{-7}\, m s^{-1}, \quad \delta \sim 100\ m$$

Thermocline scaling with wind

- with wind, thermodynamics become less important.

$$\beta v = f \frac{\partial w}{\partial z}$$

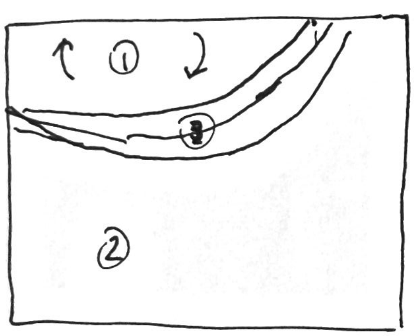
$$f \frac{\partial \vec{u}}{\partial z} = \hat{k} \times \nabla b$$

- Scaling: same, but $w \sim W_{Ek} \sim \frac{1}{\rho_0} \text{curl}\, \tau$

$$D_a = \left(\frac{W_{Ek} f L}{\beta \Delta b} \right)^{1/2} \quad \text{wind-driven circulation depth scale.}$$

$$D_a \sim 500 \text{ m}$$

The physical picture



① wind driven mixing; depth D_a

② cold abyssal flow

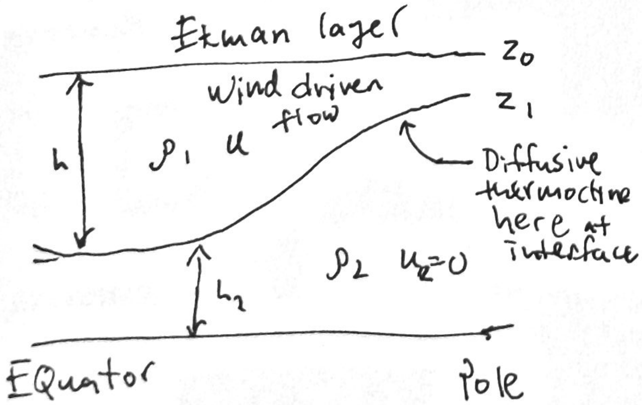
③ Internal thermocline separating abyssal flow and wind-driven circulation.

Ventilated Thermocline

Want to understand dynamics above ~~ventilated~~ internal thermocline (diffusive layer).

- wind-driven, stratified.

- Model



- Reduced gravity
- Stationary abyss
- Steady large-scale flow

$$\vec{f} \times \vec{u} = -g' \nabla h \qquad g' = g \frac{\rho_2 - \rho_1}{\rho_1}$$

$$\nabla \cdot \vec{u} = -\frac{\partial w}{\partial z}$$

$\rightsquigarrow$ vorticity $\beta v = +f \frac{\partial w}{\partial z}$

• Integrate from base to depth of ventilated thermocline

$$\beta v (z_0 - z_1) = f\left[w(z_0) - \cancel{w(z_1)} \right]$$

$w(z_0)$ given by Ekman pumping

$w(z_1) = 0$ from continuity in second layer

$$w(z_1) = \frac{\partial h_2}{\partial t} = -\nabla \cdot (u_2 h_2) = 0$$

$$\Rightarrow \beta v h = f w_{Ek}$$

$\rightsquigarrow$ geostrophy $v = \frac{g'}{f} \frac{\partial h}{\partial x}$

$$\beta \frac{g'}{f} \frac{\partial h}{\partial x} h = f w_{Ek}$$

• Integrate from x_E to x:

$$h^2 = \frac{2 f^2}{\beta g'} \int_x^{x_E} w_{Ek} \, dx' + h_E^2$$

Thermohaline Circulation and MOC

- ~~Adding to the picture of the advective diffusive balance is~~

We seek to explain the meridional overturning circulation

AMOC

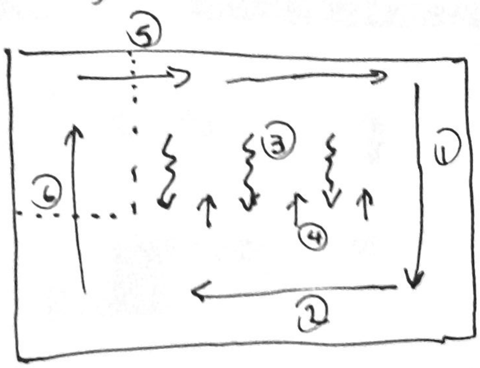
Depth

SP EQ

Why is there an overturning cell? Buoyancy-driven circulation which balances with vertical upwelling in the ~~middle~~ low latitudes is insufficient to explain why flow spreads all the way to the South

Answer has to do with wind-driven circulation

Strong winds in Southern Ocean (no continental boundaries) imply northward Ekman transport at the surface



1. Cold dense surface water is unstable and leads to downwelling
2. Cold water fills abyss
3. Surface heating warms abyssal flow
4. Some of the abyssal flow is upwelled when it is heated but rest continues downward

5. Ekman transport in Southern ocean drives return flow:

$$M_{Ek} = \frac{1}{\rho f} \vec{\tau}_s \times \hat{k}$$

$\vec{\tau}_s$ is eastward

$\vec{\tau}_s \times \hat{k}$ is southward

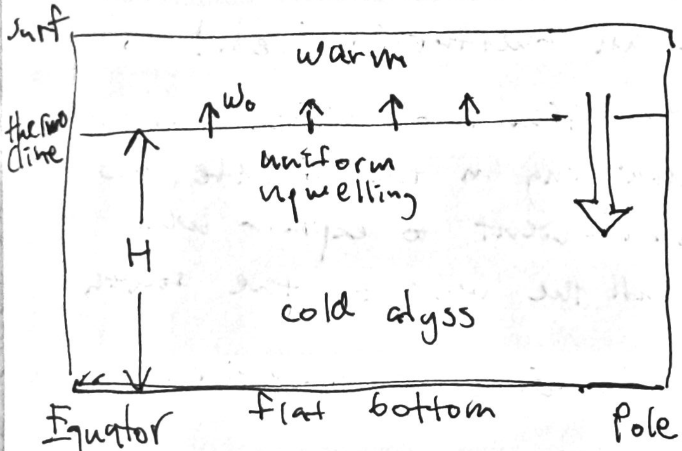
$f < 0$ so M_{Ek} is northward.

6. upwelling.

Stommel-Arons model

Simple model of the abyssal circulation.

• Model



• 1 layer homogeneous abyssal flow
• High-latitude localized forcing
• ~~Constant~~ uniform upwelling

• Equations: Planetary geostrophy on β-plane (large scale).

$$\vec{f} \times \vec{u} = -\nabla \phi$$

$$\nabla \cdot \vec{u} = -\frac{\partial w}{\partial z}$$

$$\rightsquigarrow f \frac{\partial w}{\partial z} = \beta v$$

$w > 0$ at the top of the abyss

• Integrate top to bottom

$$f\left(w_0 - \cancel{w(H)}\right) = \beta v H$$

$$\Rightarrow v = \frac{f}{\beta} \frac{w_0}{H}$$ poleward flow in the ~~abyss~~ interior

Need return flow on the boundaries.

~~From PV conservation on β-plane~~

Open questions in MOC

- Both wind-driven and buoyancy driven downwelling are important to set the MOC. How can we assess their relative importance?
- Why are there two cells in the Atlantic and only one in other basins?
  - Theories: Atlantic goes further ~~up~~ up north, mediterranean messes with stuff